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UNIT – 1

Matrices and Determinants

1. Matrices and Determinants

Example 1.1 If A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ verify that A (adj A) = (adj A) A = | A |. I_3

Solution: A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + 6\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

= 8(21 – 16) +6(– 18 +8) + 2(24 – 14)

= 8(5) +6(– 10) + 2(10)

= 40 – 60 + 20

= 60 – 60

$|A| = 0$

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{matrix} 7 & -4 & -6 & 7 \\ -4 & 3 & 2 & -4 \\ -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 2 \end{matrix}$$

Cofactor of A Aij = $\begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$

Adj A = Aij^T

Adj A = $\begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$

$$A\,(adj\,A) = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \ldots\ldots\ldots (1)$$

$$(\text{adj A})\, A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad .... \quad (2)$$

$$|\,A\,|.\,I = 0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad .... \quad (3)$$

From (1), (2) and (3)

A (adj A) = (adj A) A = | A | I_3 is verified.

Example 1.2 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non singular, find A^{-1}

Solution: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc$$

$$\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \, Adj\, A$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 1.3 Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} -5 & 1 \\ -3 & 3 \end{vmatrix} + 3\begin{vmatrix} -5 & 3 \\ -3 & 2 \end{vmatrix}$$

$$= 2(9 - 2) + 1(-15 + 3) + 3(-10 + 9)$$

$$= 2(7) + 1(-12) + 3(-1)$$

$$= 14 - 12 - 3$$

$$= 14 - 15$$

$= -1 \neq 0$, hence A^{-1} exists.

$$\begin{matrix} 3 & 1 & -5 & 3 \\ 2 & 3 & -3 & 2 \\ -1 & 3 & 2 & -1 \\ 3 & 1 & -5 & 3 \end{matrix}$$

Cofactor of A $\quad A_{ij} = \begin{bmatrix} 7 & 12 & -1 \\ 9 & 15 & -1 \\ -10 & -17 & 1 \end{bmatrix}$

$$\text{Adj } A = A_{ij}^T$$

$$= \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \, Adj\, A$$

$$= \frac{1}{-1} \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$$

$$= -1 \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}$$

Example 1.4 If A is a non-singular matrix of odd order, prove that $|adj\ A|$ is positive.

Solution: Let A be a non-singular matrix of order $2m+1$, where m = 0,1,2,3,. . .

Then, we get $|A| \neq 0$ and, by property (ii), we have $|adj\ A| = |A|^{(2m+1)-1} = |A|^{2m}$.

Since $|A|^{2m}$ is always positive, we get that $|adj\ A|$ is positive.

Example 1.5 Find a matrix A if $adj\ A = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$

Solution: $adj\ A = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$

$$|adj\ A| = \begin{vmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{vmatrix}$$

$$= 7\begin{vmatrix} 11 & 7 \\ 5 & 7 \end{vmatrix} - 7\begin{vmatrix} -1 & 7 \\ 11 & 7 \end{vmatrix} - 7\begin{vmatrix} -1 & 11 \\ 11 & 5 \end{vmatrix}$$

$$= 7(77 - 35) - 7(-7 - 77) - 7(-5 - 121)$$

$$= 7(42) - 7(-84) - 7(-126)$$

$$= 7(42 + 84 + 126)$$

$$= 7(252)$$

$$= 7(7\times 36)$$

$$= 7^2 \times 6^2$$

$$\sqrt{adj\ A} = 7 \times 6 = 42$$

We know, $A = \pm\frac{1}{\sqrt{adj\ A}}\ adj(adj\ A)$

Given: $adj\ A = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$

$$\begin{matrix} 11 & 7 & -1 & 11 \\ 5 & 7 & 11 & 5 \\ 7 & -7 & 7 & 7 \\ 11 & 7 & -1 & 11 \end{matrix}$$

Cofactor of adj A $= \begin{bmatrix} 42 & 84 & -126 \\ -84 & 126 & 42 \\ 126 & -42 & 84 \end{bmatrix}$

$$adj(adj\ A) = \begin{bmatrix} 42 & -84 & 126 \\ 84 & 126 & -42 \\ -126 & 42 & 84 \end{bmatrix}$$

$$A = \pm\frac{1}{\sqrt{adj\ A}}\ adj(adj\ A)$$

$$= \pm\frac{1}{42}\begin{bmatrix} 42 & -84 & 126 \\ 84 & 126 & -42 \\ -126 & 42 & 84 \end{bmatrix}$$

$$A = \pm\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

Example 1.6 If $adj\ A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ find A^{-1}

Solution: $adj\ A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$|adj\ A| = \begin{vmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= -1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= -1(1 - 4) - 2(1 - 4) + 2(2 - 2)$$

$$= -1(-3) - 2(-3) + 2(0)$$

$$= 3 + 6 + 0$$

$$= 9 \implies \sqrt{adj\ A} = 3$$

We know, $A^{-1} = \pm\frac{1}{\sqrt{adj\ A}}\ (adj\ A)$

Hence, $A^{-1} = \pm\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

Example1.7 If A is symmetric, prove that then adj A is also symmetric.

Solution: Given A is symmetric. $\therefore A^T = A$

By the property, $adj\ (A^T) = (adj\ A)^T$

So, $adj\ (A) = (adj\ A)^T$ hence adj A is symmetric.

Example 1.8 Verify the property $(A^T)^{-1} = (A^{-1})^T$ with A = $\begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

Solution: A = $\begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & 1 \\ 9 & 7 \end{bmatrix}$

$|A^T| = \begin{vmatrix} 2 & 1 \\ 9 & 7 \end{vmatrix}$

= 14 – 9

= 5

$\text{Adj}(A^T) = \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$

$(A^T)^{-1} = \frac{1}{|A^T|} Adj\ A^T$

$= \frac{1}{5} \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$ ....(1)

A = $\begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 9 \\ 1 & 7 \end{vmatrix}$

= 14 – 9

= 5

adj A = $\begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} Adj\ A$

$= \frac{1}{5} \begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix}$

$(A^{-1})^T = \frac{1}{5} \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$ ....(1)

From (1) and (2)

$(A^T)^{-1} = (A^{-1})^T$ is verified.

Example1.9 Verify that $(AB)^{-1} = B^{-1} A^{-1}$

with A = $\begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$

A = $\begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$

AB = $\begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$

AB = $\begin{bmatrix} 0+0 & 0+3 \\ -2+0 & -3-4 \end{bmatrix}$

AB = $\begin{bmatrix} 0 & 3 \\ -2 & -7 \end{bmatrix}$

$|AB| = \begin{vmatrix} 0 & 3 \\ -2 & -7 \end{vmatrix}$

= 0 + 6 = 6

adj (AB) = $\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{|AB|} Adj\ (AB)$

$= \frac{1}{6} \begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$ .... (1)

A = $\begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$

$|A| = \begin{vmatrix} 0 & -3 \\ 1 & 4 \end{vmatrix}$

= 0 + 3

= 3

adj A = $\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} Adj\ A$

$= \frac{1}{3} \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$

B = $\begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$

$|B| = \begin{vmatrix} -2 & -3 \\ 0 & -1 \end{vmatrix}$

= 2 +0

= 2

adj B = $\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$

$$B^{-1} = \frac{1}{|B|}\, Adj\ B$$

$$B^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -4-3 & -3+0 \\ 0+2 & 0+0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} \quad \ldots\ldots (2)$$

From (1) and (2)

$(AB)^{-1} = B^{-1}A^{-1}$ is verified.

Example 1.10 If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 + xA + yI_2 = 0_2$.
Hence, find A^{-1}.

Solution: $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 16+6 & 12+15 \\ 8+10 & 6+25 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 22 & 37 \\ 18 & 31 \end{bmatrix}$$

$$A^2 + xA + yI_2 = 0_2$$

$$\begin{bmatrix} 22 & 37 \\ 18 & 31 \end{bmatrix} + x\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 22 & 37 \\ 18 & 31 \end{bmatrix} + \begin{bmatrix} 4x & 3x \\ 2x & 5x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 22+4x+y & 37+3x+0 \\ 18+2x+0 & 31+5x+y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$22 + 4x + y = 0,\ 37 + 3x = 0,$

$18 + 2x = 0,\ 31 + 5x + y = 0$

$18 + 2x = 0$ gives

$$2x = -18$$

$$x = -\frac{18}{2}$$

$$x = -9$$

Substituting, $x = -9$ in

$$22 + 4x + y = 0$$

$$22 + 4(-9) + y = 0$$

$$22 - 36 + y = 0$$

$$-14 + y = 0$$

$$y = 14$$

Hence, Substituting $x = -9$ and $y = 14$ in

$$A^2 + xA + yI_2 = 0_2$$

$$A^2 - 9A + 14I_2 = 0_2$$

Post-multiplying this equation by A^{-1}

We get, $A - 9I + 14A^{-1} = 0_2$

$$14A^{-1} = 9I - A$$

$$A^{-1} = \frac{1}{14}(9I - A)$$

$$= \frac{1}{14}\left[9\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}\right]$$

$$= \frac{1}{14}\left[\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

Example 1.11 Prove that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Solution: Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then,

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} cos^2\theta + sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & sin^2\theta + cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly we can prove $A^T A = I$.

Hence the given matrix is orthogonal.

Example 1.12

If $A = \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}$ is orthogonal, find a, b and c and hence A^{-1}.

Solution: Given A is called orthogonal,

hence $AA^T = A^TA = I$

$$AA^T = \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix} \times \frac{1}{7}\begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix}$$

$$= \frac{1}{49}\begin{bmatrix} 36 + 9 + a^2 & 6b + 6 + 6a & 12 - 3c + 3a \\ 6b + 6 + 6a & b^2 + 4 + 36 & 2b - 2c + 18 \\ 12 - 3c + 3a & 2b - 2c + 18 & 4 + c^2 + 9 \end{bmatrix}$$

$$= \frac{1}{49}\begin{bmatrix} 45 + a^2 & 6b + 6 + 6a & 12 - 3c + 3a \\ 6b + 6 + 6a & b^2 + 40 & 2b - 2c + 18 \\ 12 - 3c + 3a & 2b - 2c + 18 & c^2 + 13 \end{bmatrix}$$

$AA^T = I$

$$\frac{1}{49}\begin{bmatrix} 45 + a^2 & 6b + 6 + 6a & 12 - 3c + 3a \\ 6b + 6 + 6a & b^2 + 40 & 2b - 2c + 18 \\ 12 - 3c + 3a & 2b - 2c + 18 & c^2 + 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 45 + a^2 & 6b + 6 + 6a & 12 - 3c + 3a \\ 6b + 6 + 6a & b^2 + 40 & 2b - 2c + 18 \\ 12 - 3c + 3a & 2b - 2c + 18 & c^2 + 13 \end{bmatrix} = 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 45 + a^2 & 6b + 6 + 6a & 12 - 3c + 3a \\ 6b + 6 + 6a & b^2 + 40 & 2b - 2c + 18 \\ 12 - 3c + 3a & 2b - 2c + 18 & c^2 + 13 \end{bmatrix} = \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

$45 + a^2 = 49$ gives
$a^2 = 49 - 45 \Longrightarrow a^2 = 4$, and

$b^2 + 40 = 49$ gives
$b^2 = 49 - 40 \Longrightarrow b^2 = 9$ and

$c^2 + 13 = 49$ gives
$c^2 = 49 - 13 \Rightarrow c^2 = 36$

$6b + 6 + 6a = 0$

$6b + 6a = -6$ gives

$b + a = -1$ ……..(1)

$12 - 3c + 3a = 0$

$-3c + 3a = -12$ gives

$-c + a = -4$ ……..(2)

$2b - 2c + 18 = 0$

$2b - 2c = -18$ gives

$b - c = -9$ ……..(3)

From (2) and (3)

$a - c = -4$

$b - c = -9$

$a - b = 5$

$a + b = -1$ ……..(1)

2a = 4 which is

a = 2

Substituting a = 2 in $a + b = -1$

$2 + b = -1$

$b = -1 - 2$

$\boldsymbol{b = -3}$

Substituting a = 2 in $a - c = -4$

$2 - c = -4$

$$-c = -4 - 2$$

$$-c = -6$$

$$\boldsymbol{c = 6}$$

$$\text{So } A = \frac{1}{7}\begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix}$$

That is $A^{-1} = A^{T}$

$$= \frac{1}{7}\begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix}$$

EXERCISE 1.1

1. Find the adjoint of the following:

(i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 2 & -4 \\ -6 & -3 \end{bmatrix}$$

(ii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Solution:

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix} \qquad \begin{array}{cccc} 4 & 1 & 3 & 4 \\ 7 & 2 & 3 & 7 \\ 3 & 1 & 2 & 3 \\ 4 & 1 & 3 & 4 \end{array}$$

$$\text{Cofactor of A Aij} = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}$$

$$\text{Adj A} = \text{Aij}^{T}$$

$$\text{Adj A} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

(iii) $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

Solution: Let $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$\begin{array}{cccc} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}$$

$$\text{Cofactor of A Aij} = \begin{bmatrix} \frac{6}{9} & \frac{6}{9} & \frac{3}{9} \\ -\frac{6}{9} & \frac{3}{9} & \frac{6}{9} \\ \frac{3}{9} & -\frac{6}{9} & \frac{6}{9} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

$$\text{Adj A} = \text{Aij}^{T}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

2. Find the inverse (if it exists) of the following:

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

Let $A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix}$$

$= 6 - 4$

$= 2 \neq 0$, hence A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj\ A$$

$$= \frac{1}{2} \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$$

(ii) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{vmatrix}$$

$$= 5 \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} + 1 \begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix}$$

$= 5(25 - 1) - 1(5 - 1) + 1(1 - 5)$

$= 5(24) - 1(4) + 1(-4)$

$= 120 - 4 - 4$

$= 120 - 8$

$= 112 \neq 0$, hence A^{-1} exists.

$$\begin{matrix} 5 & 1 & 1 & 5 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 5 & 1 & 1 & 5 \end{matrix}$$

Cofactor of A $Aij = \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$

Adj A = Aij^T

$$= \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj\ A$$

$$= \frac{1}{112} \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

$$= \frac{1(4)}{112} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{28} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

(iii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} - 3 \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix}$$

$= 2(8 - 7) - 3(6 - 3) + 1(21 - 12)$

$= 2(1) - 3(3) + 1(9)$

$= 2 - 9 + 9$

$= 2$

$= 2 \neq 0$, hence A^{-1} exists.

$$\begin{matrix} 4 & 1 & 3 & 4 \\ 7 & 2 & 3 & 7 \\ 3 & 1 & 2 & 3 \\ 4 & 1 & 3 & 4 \end{matrix}$$

Cofactor of A $Aij = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}$

Adj A = Aij^T

$$= \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$A^{-1} = \frac{1}{|A|}\, Adj\ A$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

3. If $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$,

show that $[F(\alpha)]^{-1} = F(-\alpha)$

Solution: $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$

$$|F(\alpha)| = \begin{vmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{vmatrix}$$

$$= \cos\alpha \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ -\sin\alpha & \cos\alpha \end{vmatrix}$$

$$+\sin\alpha \begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix}$$

$= \cos\alpha\,(\cos\alpha) + \sin\alpha\,(\sin\alpha)$

$= cos^2\alpha + sin^2\alpha$

$= 1 \neq 0$, hence $F(\alpha)^{-1}$ exists.

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & \cos\alpha & -sin\alpha & 0 \\ 0 & \sin\alpha & cos\alpha & 0 \\ 1 & 0 & 0 & 1 \end{matrix}$$

Cofactor of $F(\alpha)$

$$= \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & cos^2\alpha + sin^2\alpha & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

Adj $F(\alpha) = F(\alpha)^T$

$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$F(\alpha)^{-1} = \frac{1}{|F(\alpha)|}\, Adj\ F(\alpha)$$

$$= \frac{1}{1}\begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$F(\alpha)^{-1} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \text{......(1)}$$

$$F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix}$$

We know $\cos(-\alpha) = \cos\alpha$

and $\sin(-\alpha) = -\sin\alpha$

$$F(-\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \text{......(1)}$$

From (1) *and* (2)

$[F(\alpha)]^{-1} = F(-\alpha)$, hence proved.

4. If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7I_2 = 0_2$. Hence find A^{-1}.

Solution: $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$A^2 - 3A - 7I_2 =$

$$\begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2 \text{ Hence Proved.}$$

So, $A^2 - 3A - 7I_2 = 0_2$

Post-multiplying this equation by A^{-1}

We get, $A - 3I - 7A^{-1} = 0_2$

$$A - 3I = 7A^{-1}$$

$$7A^{-1} = A - 3I$$

$$A^{-1} = \frac{1}{7}(A - 3I)$$

$$= \frac{1}{7}\left[\begin{pmatrix} 5 & 3 \\ -1 & -2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{7}\left[\begin{pmatrix} 5 & 3 \\ -1 & -2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right]$$

$$= \frac{1}{7}\left[\begin{pmatrix} 5-3 & 3-0 \\ -1-0 & -2-3 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

5. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$ *prove that* $A^{-1} = A^T$

Solution: $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$

$$AA^T = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}\frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32+28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$= \frac{81}{81}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{I.}$$

That is $AA^T = I$ hence A is orthogonal.

Therfore $A^{-1} = A^T$

6. If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$ verify that

A (adj A) = (adj A) A = | A |. I_2

Solution: $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 8 & -4 \\ -5 & 3 \end{vmatrix}$$

$$= 24 - 20$$

$$|A| = 4$$

$$\text{Adj A} = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$\text{A (adj A)} = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{.........} \quad (1)$$

$$\text{(adj A) A} = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{.........} \quad (1)$$

$$\text{| A |. I} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ .... (3)

From (1), (2) and (3)

A (adj A) = (adj A) A = | A | I_2 is verified.

7. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

Verify that $(AB)^{-1} = B^{-1} A^{-1}$

$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} -3+10 & -9+4 \\ -7+25 & -21+10 \end{bmatrix}$

$AB = \begin{bmatrix} 7 & -5 \\ 18 & -11 \end{bmatrix}$

$|AB| = \begin{vmatrix} 7 & -5 \\ 18 & -11 \end{vmatrix}$

$= -77 + 90 = 13$

$adj\,(AB) = \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{|AB|} Adj\,(AB)$

$= \frac{1}{13} \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$ .... (1)

$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix}$

$= 15 - 14$

$= 1$

$adj\,A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} Adj\,A$

$= \frac{1}{1} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$|B| = \begin{vmatrix} -1 & -3 \\ 5 & 2 \end{vmatrix}$

$= -2 + 15$

$= 13$

$adj\,B = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|} Adj\,B$

$B^{-1} = \frac{1}{13} \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

$B^{-1} A^{-1} = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix} \frac{1}{13} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$= \frac{1}{13} \begin{bmatrix} 10-21 & -4+9 \\ -25+7 & 10-3 \end{bmatrix}$

$= \frac{1}{13} \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$ ...... (2)

From (1) and (2)

$(AB)^{-1} = B^{-1} A^{-1}$ is verified.

8. If $adj\,A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$ Find A

Solution: $adj\,A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$

$$|adj\ A| = \begin{vmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 12 & -7 \\ 0 & 2 \end{vmatrix} +4\begin{vmatrix} -3 & -7 \\ -2 & 2 \end{vmatrix} + 2\begin{vmatrix} -3 & 12 \\ -2 & 0 \end{vmatrix}$$

$= 2(24 – 0) +4 (– 6 –14) + 2(0+24)$

$= 2(24) +4 (–20) + 2(24)$

$= 48 – 80 + 48 = 96 – 80$

$= 16 \Rightarrow \sqrt{adj\ A} = 4$

We know, A = $\pm\frac{1}{\sqrt{adj\ A}}\ adj(adj\ A)$

Given: $adj\ A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$

$$\begin{matrix} 12 & -7 & -3 & 12 \\ 0 & 2 & -2 & 0 \\ -4 & 2 & 2 & -4 \\ 12 & -7 & -3 & 12 \end{matrix}$$

Cofactor of adj A = $\begin{bmatrix} 24 & 20 & 24 \\ 8 & 8 & 8 \\ 4 & 8 & 12 \end{bmatrix}$

$adj(adj\ A) = \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$

A = $\pm\frac{1}{\sqrt{adj\ A}}\ adj(adj\ A)$

$$= \pm\frac{1}{4}\begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$A = \pm\ \frac{1}{4}\ (4)\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

$$A = \pm\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

9. If $adj\ A = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$ find A^{-1}

Solution: $adj\ A = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$

$$|adj\ A| = \begin{vmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{vmatrix}$$

$$= 0\begin{vmatrix} 2 & -6 \\ 0 & 6 \end{vmatrix} +2\begin{vmatrix} 6 & -6 \\ -3 & 6 \end{vmatrix} + 0\begin{vmatrix} 6 & 2 \\ -3 & 0 \end{vmatrix}$$

$= 0 + 2\ (36 – 18) + 0$

$= 2\ (36 – 18)$

$= 2\ (18)$

$= 36$

$\sqrt{adj\ A} = 6$

We know, $A^{-1} = \pm\frac{1}{\sqrt{adj\ A}}\ (adj\ A)$

Hence, $A^{-1} = \pm\frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$

10. Find $adj(adj\ A)$ if $adj\ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Given: $adj\ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$\begin{matrix} 2 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{matrix}$$

Cofactor of adj A = $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

$adj(adj\ A)$ = (Cofactor of adj A $)^T$

$adj(adj\ A) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

11. If A = $\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ show that

$A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

Solution: A = $\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix}$

$= 1 + tan^2 x$

$= sec^2 x$

adj A = $\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

We know, $A^{-1} = \frac{1}{|A|}\ Adj\ A$

$= \frac{1}{sec^2 x} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$= cos^2 x \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$= cos^2 x \begin{bmatrix} 1 & -\frac{\sin x}{\cos x} \\ \frac{\sin x}{\cos x} & 1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} cos^2 x & -\cos x \sin x \\ \cos x \sin x & cos^2 x \end{bmatrix}$

A = $\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & -\frac{\sin x}{\cos x} \\ \frac{\sin x}{\cos x} & 1 \end{bmatrix}$

$A^T A^{-1}$

$= \begin{bmatrix} 1 & -\frac{\sin x}{\cos x} \\ \frac{\sin x}{\cos x} & 1 \end{bmatrix} \begin{bmatrix} cos^2 x & -\cos x \sin x \\ \cos x \sin x & cos^2 x \end{bmatrix}$

$= \begin{bmatrix} cos^2 x - sin^2 x & -2\sin x \cos x \\ 2\sin x \cos x & cos^2 x - sin^2 x \end{bmatrix}$

$= \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

Since $cos^2 x - sin^2 x = \cos 2x$ and

$2\sin x \cos x = \sin 2x$

12. Find the matrix A for which

$A \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

Solution: $A \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

Let A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$AB = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$ gives

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

$\begin{bmatrix} 5a - b & 3a - 2b \\ 5c - d & 3c - 2d \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

$5a - b = 14$ …… (1)

$3a - 2b = 7$

$5c - d = 7$

$3c - 2d = 7$

Solving $(1) \times 2$

$10a - 2b = 28$

$3a - 2b = 7$

$7a = 21 \Longrightarrow a = 3$

Substituting $a = 3\ in\ (1)$

$5a - b = 14$

$5(3) - b = 14$

$15 - b = 14$

$-b = 14 - 15$

$-b = -1 \Rightarrow b = 1$

Solving $(3) \times 2$

$10c - 2d = 14$

$3c - 2d = 7$

$7c = 7 \Longrightarrow c = 1$

Substituting $c = 1$ *in* (3)

$5c - d = 7$

$5(1) - d = 7$

$5 - d = 7$

$-d = 7 - 5$

$-d = 2 \Rightarrow d = -2$

Hence A $= \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$

13. Given A=$\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, B=$\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$and C=$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ find a matrix X such that AXB=C

Solution: AXB=C

Let X=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a-c & b-d \\ 2a & 2b \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3(a-c)+1(b-d) & -2(a-c)+1(b-d) \\ 3(2a)+1(2b) & -2(2a)+1(2b) \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3a-3c+b-d & -2a+2c+b-d \\ 6a+2b & -4a+2b \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$6a + 2b = 2$, and $-4a + 2b = 2$

$3a - 3c + b - d = 1$ and

$-2a + 2c + b - d = 1$

Solving $6a + 2b = 2$

$-4a + 2b = 2$

$10a = 0$ gives $a = 0$

Substituting $a = 0$ in

$-4a + 2b = 2$

$2b = 2$ gives $b = 1$

Substituting $a = 0$ and $b = 1$ in

$3a - 3c + b - d = 1$

$0 - 3c + 1 - d = 1$

$-3c + 1 - d = 1$

$-3c - d = 1 - 1$

$-3c - d = 0$ and

$-2a + 2c + b - d = 1$

$0 + 2c + 1 - d = 1$

$2c + 1 - d = 1$

$2c - d = 1 - 1$

$2c - d = 0$

Solving $-3c - d = 0$ and

$2c - d = 0$ we get

$c = 0$

Substituting $c = 0$ in

$2c - d = 0$

$d = 0$

Substituting $a = 0$, $b = 1$, $c = 0$ and $d = 0$

in X matrix

X=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we get

X=$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

14. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$,

show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$

Solution: Given $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 0\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$= 0 - 1(0 - 1) + 1(1 - 0)$

$= -1(-1) + 1(1)$

$= 1 + 1$

$|A| = 2 \neq 0$, hence A^{-1} exists.

0	1	1	0
1	0	1	1
1	1	0	1
0	1	1	0

Cofactor of A Aij = $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Adj A = Aij^T

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$A^{-1} = \frac{1}{|A|} \, Adj\ A$

$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ …… (1)

$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^2 - 3I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 1-0 & 1-0 \\ 1-0 & 2-3 & 1-0 \\ 1-0 & 1-0 & 2-3 \end{bmatrix}$$

$\frac{1}{2}(A^2 - 3I) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ …… (2)

From (1)and (2) $A^{-1} = \frac{1}{2}(A^2 - 3I), proved.$

15. Decrypt the received encoded message $[2 \quad -3][20 \quad 4]$with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes are described by the numbers 1−26 to the letters A−Z respectively, and the number 0 to a blank space.

Solution: Given Encoding matrix

$$A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix}$$

$= -1 + 2$

$= 1$

$$\text{adj } A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \, Adj\ A$$

$$= \frac{1}{1} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

Decoding matrix

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

Given encoded message is

$[2 \quad -3], [20 \quad 4]$

Coded row matrix	Decoding matrix	Decoded row matrix

$$[2 \quad -3] \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = [2+6 \quad 2+3]$$

$$[20 \quad 4] \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = [20-8 \quad 20-4]$$

So, the sequence of decoded row matrices is

$[8 \quad 5][12 \quad 16]$. Thus, the receiver reads the message as **"HELP"**.

Example 1.13 Reduce the matrix

$\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ to a row – echelon form.

$$\text{Let } A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 = R_2 + 2R_1 \\ R_3 = R_3 + R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} R_3 = 2R_3 - R_2$$

$$\rightarrow \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

This is also a row-echelon form of the given matrix.

Example 1.14 Reduce the matrix

$\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$ to a row – echelon form.

$$\text{Let } A = \begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & 6 \\ 2 & 0 & -1 & 5 \\ 0 & 2 & 4 & 0 \end{bmatrix} C_1 \leftrightarrow C_3$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & 6 \\ 0 & 6 & 1 & 7 \\ 0 & 2 & 4 & 0 \end{bmatrix} R_2 = R_2 - 2R_1$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & 6 \\ 0 & 6 & 1 & 7 \\ 0 & 0 & 11 & -7 \end{bmatrix} R_3 = 3R_3 - R_2$$

This is also a row-echelon form of the given matrix.

Example 1.15

Find the rank of each of the following matrices: (i) $\begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 5 \\ 3 & 3 & 6 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ R_2 \\ R_3 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$$

The Matrix is in Echelon form.

The number of non zero rows are 2.

$\therefore \ \rho(A) = 2$

(ii) $\begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$

Let A $= \begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 3 & 4 & -2 \\ -2 & -1 & -3 & 4 \\ -1 & 7 & 6 & 2 \end{bmatrix} C_1 \leftrightarrow C_3$

$\rightarrow \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & 5 & 5 & 0 \\ -1 & 7 & 6 & 2 \end{bmatrix} R_2 = R_2 + 2R_1$

$\rightarrow \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & 5 & 5 & 0 \\ 0 & 10 & 10 & 0 \end{bmatrix} R_3 = R_3 + R_1$

$\rightarrow \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 = R_3 - 2R_2$

The Matrix is in Echelon form.

The number of non zero rows are 2.

$\therefore\ \rho(A) = 2$

Example 1.16 Find the rank of the following matrices which are in row-echelon form:

(i) $\begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

The number of non zero rows are 3.

$\therefore\ \rho(A) = 3$

(ii) $\begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The number of non zero rows are 2.

$\therefore\ \rho(A) = 2$

(iii) $\begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The number of non zero rows are 2.

$\therefore\ \rho(A) = 2$

Example 1.17

Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

by reducing it to a row-echelon form.

Let A $= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$

$\rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 = R_3 - 2R_2 \end{matrix}$

The Matrix is in Echelon form.

The number of non zero rows are 2.

$\therefore\ \rho(A) = 2$

Example 1.18 Find the rank of the matrix $\begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$ by reducing it to a row-echelon form.

Let A $= \begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & -2 & 8 & 7 \\ 6 & 2 & -1 & 7 \end{bmatrix} R_2 = 2R_2 + 3R_1$

$\rightarrow \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & -2 & 8 & 7 \\ 0 & 8 & -13 & -2 \end{bmatrix} R_3 = R_3 - 3R_1$

$\rightarrow \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & -2 & 8 & 7 \\ 0 & 0 & 19 & 26 \end{bmatrix} R_3 = R_3 + 4R_2$

The no. of non zero rows are 3. $\therefore\ \rho(A) = 3$

Example 1.19 Show that the matrix

$\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$ is non-singular and reduce it to the identity matrix by elementary row transformations.

Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$

$$\to \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix} \begin{matrix} R_1 = R_1 \div 3 \\ R_2 \\ R_3 \end{matrix}$$

$$\to \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & -\frac{2}{3} & -\frac{11}{3} \\ 0 & \frac{1}{3} & -\frac{17}{3} \end{bmatrix} \begin{matrix} R_1 \\ R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 5R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & \frac{11}{2} \\ 0 & \frac{1}{3} & -\frac{17}{3} \end{bmatrix} \begin{matrix} R_1 \\ R_2 = R_2\left(-\frac{3}{2}\right) \\ R_3 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{11}{2} \\ 0 & 0 & -\frac{15}{2} \end{bmatrix} \begin{matrix} R_1 = R_1 - \frac{1}{3}R_2 \\ R_2 \\ R_3 = R_3 - \frac{1}{3}R_2 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{11}{2} \\ 0 & 0 & -\frac{15}{2} \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 = R_3\left(-\frac{2}{15}\right) \end{matrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 = R_1 + \frac{1}{2}R_3 \\ R_2 = R_2 - \frac{11}{2}R_3 \\ R_3 \end{matrix}$$

Example 1.20 Find the inverse of the non-singular matrix $A = \begin{bmatrix} 0 & 5 \\ -1 & 6 \end{bmatrix}$, by Gauss-Jordan method.

Solution: $(A|I_2) = \left(\begin{array}{cc|cc} 0 & 5 & 1 & 0 \\ -1 & 6 & 0 & 1 \end{array}\right)$

$$\to \left(\begin{array}{cc|cc} -1 & 6 & 0 & 1 \\ 0 & 5 & 1 & 0 \end{array}\right) R_1 \leftrightarrow R_2$$

$$\to \left(\begin{array}{cc|cc} 1 & -6 & 0 & -1 \\ 0 & 5 & 1 & 0 \end{array}\right) \begin{matrix} R_1 = R_1(-1) \\ R_2 \end{matrix}$$

$$\to \left(\begin{array}{cc|cc} 1 & -6 & 0 & -1 \\ 0 & 1 & \frac{1}{5} & 0 \end{array}\right) \begin{matrix} R_1 \\ R_2 = \frac{1}{5}R_2 \end{matrix}$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & \frac{6}{5} & -1 \\ 0 & 1 & \frac{1}{5} & 0 \end{array}\right) \begin{matrix} R_1 = R_1 + 6R_2 \\ R_2 \end{matrix}$$

Hence $A^{-1} = \begin{bmatrix} \frac{6}{5} & -1 \\ \frac{1}{5} & 0 \end{bmatrix}$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}$$

Example 1.21 Find the inverse of $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by Gauss-Jordan method.

Solution: $(A|I_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$

$$\to \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) R_1 = R_1 \div 2$$

$$\to \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \begin{matrix} R_1 \\ R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 2R_1 \end{matrix}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) R_2 = 2R_2$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) R_1 = R_1 - \frac{1}{2}R_2$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \begin{matrix} R_1 = R_1 - R_3 \\ R_2 = R_2 + R_3 \end{matrix}$$

Hence $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

EXERCISE 1.2

1. Find the rank of the following matrices by minor method:

(i) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

Let A = $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix}$

$= 4 - 4$

$= 0$

$\therefore\ \rho(A) = 1$

(ii) $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

Let A = $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

A is a matrix order of 3×2,

$\therefore\ \rho(A) \leq 2$

We find that there is a second order minor,

$\begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix}$

$= 7 - 12$

$= -5 \neq 0$

$\therefore\ \rho(A) = 2$

(iii) $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

Let A = $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

A is a matrix order of 2×4,

$\therefore\ \rho(A) \leq 2$

We find that there is a second order minor,

$\begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix}$

$= -1 + 0$

$= -1 \neq 0$

$\therefore\ \rho(A) = 2$

(iv) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

Let A = $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

A is a matrix order of 3×3,

$\therefore\ \rho(A) \leq 3$

$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix}$

$= 1\begin{vmatrix} 4 & -6 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -6 \\ 5 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix}$

$= 1(-4+6) + 2(-2+30) + 3(2-20)$

$= 1(2) + 2(28) + 3(-18)$

$= 2 + 56 - 54$

$= 58 - 54$

$= 4$

$|A| = 4 \neq 0$,

$\therefore\ \rho(A) = 3$

(v) $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

Let A = $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

A is a matrix order of 3× 4,

$\therefore\ \rho(A) \leq 3$

We find that there is a third order minor,

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 0 & 2 \end{vmatrix}$$

$= 1\begin{vmatrix} 4 & 3 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix}$

$= 1(8 - 0) - 2(4 - 3) + 1(0 - 4)$

$= 1(8) - 2(1) + 1(-4)$

$= 8 - 2 - 4$

$= 8 - 6$

$= 2 \neq 0,$

$\therefore\ \rho(A) = 3$

2. Find the rank of the following matrices by row reduction method:

(i) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

Let A = $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 5R_1 \end{matrix}$

$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 = R_3 - 2R_2 \end{matrix}$

The Matrix is in Echelon form.

The number of non zero rows are 2.

$\therefore\ \rho(A) = 2$

(ii) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

Let A = $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 = 7R_3 - 4R_2 \\ R_4 = 7R_4 - 3R_2 \end{matrix}$

$\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 = 8R_4 - R_3 \end{matrix}$

The Matrix is in Echelon form.

The number of non zero rows are 3.

$\therefore\ \rho(A) = 3$

(iii) $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

Let A = $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 3 & -8 & 5 & 2 \\ 0 & 1 & -7 & 8 \\ -1 & 2 & 3 & -2 \end{bmatrix} R_2 = 3R_2 - 2R_1$

$\rightarrow \begin{bmatrix} 3 & -8 & 5 & 2 \\ 0 & 1 & -7 & 8 \\ 0 & -2 & 14 & -4 \end{bmatrix} R_3 = 3R_3 + R_1$

$\rightarrow \begin{bmatrix} 3 & -8 & 5 & 2 \\ 0 & 1 & -7 & 8 \\ 0 & 0 & 0 & 12 \end{bmatrix} R_3 = R_3 + 2R_2$

The Matrix is in Echelon form, since the number of non zero rows are 3, $\rho(A) = 3$

3. Find the inverse of each of the following by Gauss-Jordan method:

(i) $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

Let A = $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

Solution: $(A|I_2) = \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array}\right)$

$\rightarrow \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 5 & -2 & 0 & 1 \end{array}\right) \begin{matrix} R_1 = R_1 \div 2 \\ R_2 \end{matrix}$

$\rightarrow \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right) \begin{matrix} R_1 \\ R_2 = R_2 - 5R_1 \end{matrix}$

$\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right) \begin{matrix} R_1 = R_1 + R_2 \\ R_2 \end{matrix}$

$\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right) \begin{matrix} R_1 \\ R_2 = R_2 \times 2 \end{matrix}$

Hence $A^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

Solution: $(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right) \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - 4R_1 \end{matrix}$

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right) \begin{matrix} R_2 \\ R_3 = R_3 - 4R_2 \end{matrix}$

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right) \begin{matrix} R_2 = R_2 + R_1 \\ R_3 \end{matrix}$

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right) R_1 = R_1 + R_2$

Hence $A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Solution: $(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - R_1 \end{matrix}$

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) R_3 = R_3 + 2R_2$

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) R_3 = R_3(-1)$

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) R_2 = R_2 + 3R_3$

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) R_1 = R_1 - 3R_3$

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) R_1 = R_1 - 2R_2$

Hence $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

Example 1.22 Solve the following system of linear equations, using matrix inversion method: $5x + 2y = 3, 3x + 2y = 5$.

Solution: $5x + 2y = 3, 3x + 2y = 5$.

A = $\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$ X = $\begin{bmatrix} x \\ y \end{bmatrix}$ B = $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix}$

= 10 – 6 = 4

adj A $= \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$

A^{-1} $= \frac{1}{|A|}\, Adj\ A$

$= \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$

A^{-1} $= \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$

$X = A^{-1} \times B$

$= \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 \\ 16 \end{bmatrix}$

(x, y) = (−1,4)

Example 1.23 Solve the following system of equations, using matrix inversion method: $2x_1 + 3x_2 + 3x_3 = 5, x_1 - 2x_2 + x_3 = -4$

and $3x_1 - x_2 - 2x_3 = 3$

Solution: $2x_1 + 3x_2 + 3x_3 = 5$

$x_1 - 2x_2 + x_3 = -4$

$3x_1 - x_2 - 2x_3 = 3$

A $= \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$ X $= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ B $= \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix}$

$= 2 \begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} + 3 \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}$

$= 2(4 + 1) - 3(-2 - 3) + 3(-1 + 6)$

$= 2(5) - 3(-5) + 3(5)$

$= 10 + 15 + 15 = 40$

A $= \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$ $\quad \begin{matrix} -2 & 1 & 1 & -2 \\ -1 & -2 & 3 & -1 \\ 3 & 3 & 2 & 3 \\ -2 & 1 & 1 & -2 \end{matrix}$

Cofactor of A Aij $= \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}$

Adj A = Aij^T

Adj A $= \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$

A^{-1} $= \frac{1}{|A|}\, Adj\ A$

$= \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$

$X = A^{-1} \times B$

$= \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$= \frac{1}{40} \begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix}$

$= \frac{1}{40} \begin{bmatrix} 52 - 12 \\ 80 \\ 25 - 65 \end{bmatrix}$

$= \frac{1}{40} \begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$(x_1, x_2, x_3) = (1, 2, -1)$

Example 1.24

If $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and

$B=\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}$, find the products

AB and BA and hence solve the system of equations $x-y+z=4, x-2y-2z=9$ and $2x+y+3z=1$.

Solution:

$$A=\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}, B=\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}$$

$$AB=\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-4+4+8 & 4-8+4 & -4-8+12\\ -7+1+6 & 7-2+3 & -7-2+9\\ 5-3-2 & -5+6-1 & 5+6-3\end{bmatrix}$$

$$=\begin{bmatrix}12-4 & 8-8 & 12-12\\ -7+7 & 10-2 & -9+9\\ 5-5 & -6+6 & 11-3\end{bmatrix}$$

$$AB=\begin{bmatrix}8 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 8\end{bmatrix}$$

$$\text{and } BA=\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}$$

$$=\begin{bmatrix}-4+7+5 & 4-1-3 & 4-3-1\\ -4+14-10 & 4-2+6 & 4-6+2\\ -8-7+15 & 8+1-9 & 8+3-3\end{bmatrix}$$

$$=\begin{bmatrix}12-4 & 4-4 & 4-4\\ -14+14 & 6-6 & 6-6\\ 15-15 & 9-9 & 11-3\end{bmatrix}$$

$$BA=\begin{bmatrix}8 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 8\end{bmatrix}$$

So, we get $AB = BA = 8I_3$.

That is $\left(\frac{1}{8}A\right)B = B\left(\frac{1}{8}A\right) = I_3$.

$\therefore B^{-1} = \frac{1}{8}(A)$

Writing the given system of equations in matrix form, we get

$$\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 9\\ 1\end{bmatrix}$$

$$\text{So } \begin{bmatrix}x\\ y\\ z\end{bmatrix}=B^{-1}\begin{bmatrix}4\\ 9\\ 1\end{bmatrix}$$

$$=\frac{1}{8}(A)\begin{bmatrix}4\\ 9\\ 1\end{bmatrix}$$

$$=\frac{1}{8}\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}\begin{bmatrix}4\\ 9\\ 1\end{bmatrix}$$

$$=\frac{1}{8}\begin{bmatrix}-16+36+4\\ -28+9+3\\ 20-27-1\end{bmatrix}$$

$$=\frac{1}{8}\begin{bmatrix}-16+40\\ -28+12\\ 20-28\end{bmatrix}$$

$$=\frac{1}{8}\begin{bmatrix}24\\ -16\\ -8\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ -2\\ -1\end{bmatrix}$$

EXERCISE 1.3

1. Solve the following system of linear equations by matrix inversion method:

(i) $2x+5y=-2, \quad x+2y=-3$.

Solution: $2x+5y=-2, x+2y=-3$.

$A=\begin{bmatrix}2 & 5\\ 1 & 2\end{bmatrix}$ $X=\begin{bmatrix}x\\ y\end{bmatrix}$ $B=\begin{bmatrix}-2\\ -3\end{bmatrix}$

$$|A|=\begin{vmatrix}2 & 5\\ 1 & 2\end{vmatrix}$$

$= 4 - 5$

$= - 1$

adj A = $\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$

A^{-1} = $\frac{1}{|A|}$ $Adj\ A$

= $\frac{1}{-1}$ $\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$

A^{-1} = $-1\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$

$X = A^{-1} \times B$

$= -1\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} -4+15 \\ 2-6 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} 11 \\ -4 \end{bmatrix}$

(x, y) = $(-11, 4)$

(ii) $2x - y = 8,\ 3x + 2y = -2$.

Solution: $2x - y = 8,\ 3x + 2y = -2$.

A = $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ X = $\begin{bmatrix} x \\ y \end{bmatrix}$ B = $\begin{bmatrix} 8 \\ -2 \end{bmatrix}$

$|A|$= $\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$

= 4 +3

= 7

adj A = $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

A^{-1} = $\frac{1}{|A|}$ $Adj\ A$

= $\frac{1}{7}$ $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

A^{-1} = $\frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

$X = A^{-1} \times B$

$= \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -2 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 16-2 \\ -24-4 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 14 \\ -28 \end{bmatrix}$

(x, y) = $(2, -4)$

(iii) $2x + 3y - z = 9, x + y + z = 9$
and $3x - y - z = -1$.
Solution:

A = $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$ X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ B = $\begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix}$

$= 2\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}$

= 2(– 1+1) –3(– 1 – 3) – 1(– 1– 3)

= 2(0) –3(– 4) – 1(– 4)

= 0 +12 +4

= 16

A = $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$

$\begin{matrix} 1 & 1 & 1 & 1 \\ -1 & -2 & 3 & -1 \\ 3 & -1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{matrix}$

Cofactor of A Aij = $\begin{bmatrix} 0 & 4 & -4 \\ 4 & 1 & 11 \\ 4 & -3 & -1 \end{bmatrix}$

Adj A = Aij^{T}

Adj A = $\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$

A^{-1} = $\frac{1}{|A|}$ $Adj\ A$

$= \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$

$$X = A^{-1} \times B$$

$$= \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 0+36-4 \\ 36+9+3 \\ -36+99+1 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 36-4 \\ 48 \\ -36+100 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 32 \\ 48 \\ 64 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

(x, y, z) = (2, 3, 4)

(iv) $x + y + z - 2 = 0,$
$6x - 4y + 5z - 31 = 0$
$5x + 2y + 2z = 13$

Solution: $x + y + z = 2,$
$6x - 4y + 5z = 31$
$5x + 2y + 2z = 13$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} -4 & 5 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 6 & 5 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} 6 & -4 \\ 5 & 2 \end{vmatrix}$$

$= 1(-8 - 10) - 1(12 - 25) + 1(12 + 20)$

$= 1(-18) - 1(-13) + 1(32)$

$= -18 + 13 + 32$

$= -18 + 45$

$= 27$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix}$$

$$\begin{matrix} -4 & 5 & 6 & -4 \\ 2 & 2 & 5 & 2 \\ 1 & 1 & 1 & 1 \\ -4 & 5 & 6 & -4 \end{matrix}$$

$$\text{Cofactor of A Aij} = \begin{bmatrix} -18 & 13 & 32 \\ 0 & -3 & 3 \\ 9 & 1 & -10 \end{bmatrix}$$

Adj A = Aij^T

$$\text{Adj A} = \begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj\ A$$

$$= \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$X = A^{-1} \times B$$

$$= \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}\begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} -36+0+117 \\ 26-93+13 \\ 64+93-130 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} 117-36 \\ 39-93 \\ 157-130 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} 81 \\ -54 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \rightarrow (x, y, z) = (3, -2, 1)$$

2.If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x + y + 2z = 1, 3x + 2y + z = 7$ and $2x + y + 3z = 2.$

Solution:

$$A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+9 & -5+5 & -10+10 \\ 10-10 & 9-5 & 15-15 \\ 3-3 & 2-2 & 3-3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\text{and } BA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5+7+2 & 1+1-2 & 3-5+2 \\ -15+14+1 & 3+2-1 & 9-10+1 \\ -10+7+3 & 2+1-3 & 6-5+3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+9 & 2-2 & 5-5 \\ -15+15 & 5-1 & 10-10 \\ -10+10 & 3-3 & 9-5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

So, we get $AB = BA = 4I_3$.

That is $\left(\frac{1}{4}A\right)B = B\left(\frac{1}{4}A\right) = I_3$.

$$\therefore B^{-1} = \frac{1}{4}(A)$$

Writing the given system of equations in matrix form, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$\text{So } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4}(A) \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -5+7+6 \\ 7+7-10 \\ 1-7+2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -5+13 \\ 14-10 \\ 3-7 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was Rs. 19,800 per month at the end of the first month after 3 years of service and Rs. 23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.)

Solution:

Let the monthly salary be Rs. x

Let the annual increment be Rs. y

Then from the given data

$$x + 3y = 19{,}800$$

$$x + 9y = 23{,}400$$

Solving by matrix inversion method,

$A = \begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix}$

$= 9 - 3$

$= 6$

adj $A = \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\, Adj\ A$

$= \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$

$X = A^{-1} \times B$

$= \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 19{,}800 \\ 23{,}400 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1{,}78{,}200 - 70{,}200 \\ -19{,}800 + 23{,}400 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1{,}08{,}000 \\ 3{,}600 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18{,}000 \\ 600 \end{bmatrix}$

Hence,

Monthly salary $x =$ Rs.18,000

Annual increment $y =$ Rs.600

4. 4 men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

Solution:

Let 1 man can do the work in $x$days

In 1 day he completes $\frac{1}{x}$ of the work

Let 1 woman can do the work in y days

In 1 day she completes $\frac{1}{y}$ of the work

4 men and 4 women can finish $\frac{1}{3}$ piece of work jointly in 1day is $\frac{4}{x} + \frac{4}{y} = \frac{1}{3}$

2 men and 5 women can finish $\frac{1}{4}$ piece of work jointly in 1day is $\frac{2}{x} + \frac{5}{y} = \frac{1}{4}$

Solving $\frac{4}{x} + \frac{4}{y} = \frac{1}{3}$ and $\frac{2}{x} + \frac{5}{y} = \frac{1}{4}$

Let $\frac{1}{x} = a$ and $\frac{1}{y} = b$ then

$4a + 4b = \frac{1}{3}$

$2a + 5b = \frac{1}{4}$

$A = \begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix}$ $X = \begin{bmatrix} a \\ b \end{bmatrix}$ $B = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$

$|A| = \begin{vmatrix} 4 & 4 \\ 2 & 5 \end{vmatrix}$

$= 20 - 8$

$= 12$

adj $A = \begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\, Adj\ A$

$= \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$

$X = A^{-1} \times B$

$$= \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{12}\begin{bmatrix} \frac{5}{3} - \frac{4}{4} \\ -\frac{2}{3} + \frac{4}{4} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{12}\begin{bmatrix} \frac{20-12}{12} \\ \frac{-8+12}{12} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{12}\begin{bmatrix} \frac{8}{12} \\ \frac{4}{12} \end{bmatrix}$$

$$\text{a} = \frac{1}{12} \times \frac{8}{12} = \frac{1}{12} \times \frac{2}{3} = \frac{1}{6 \times 3} = = \frac{1}{18}$$

$a = \frac{1}{x} = \frac{1}{18}$ gives $x = 18$

$$\text{b} = \frac{1}{12} \times \frac{4}{12} = \frac{1}{12} \times \frac{1}{3} = \frac{1}{12 \times 3} = = \frac{1}{36}$$

$b = \frac{1}{y} = \frac{1}{18}$ gives $y = 36$

1 man can do the work in $x = 18$ days

1 woman can do the work in $y = 36$ days

5. The prices of three commodities A, B and C are Rs x, y, and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C. In the process, P, Q and R earn Rs 15,000, Rs 1,000 and Rs 4,000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem.)

Solution: Given the prices of three commodities A, B and C are Rs x, y, and z per units.

From the data given,

$$-4y + 2x + 5z = 15{,}000$$
$$-2z + 3x + y = 1{,}000$$
$$\text{and } -x + 3y + z = 4{,}000$$

In standard form

$$2x - 4y + 5z = 15{,}000$$
$$3x + y - 2z = 1{,}000$$
$$-x + 3y + z = 4{,}000$$

$$\text{A} = \begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix} \text{ X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ B} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix}$$

$$= 2(1+6) + 4\,(3-2) + 5\,(9+1)$$

$$= 2(7) + 4(1) + 5(10)$$

$$= 14 + 4 + 50$$

$$= 68$$

$$\text{A} = \begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix} \quad \begin{matrix} 1 & -2 & 3 & 1 \\ 3 & 1 & -1 & 3 \\ -4 & 5 & 2 & -4 \\ 1 & -2 & 3 & 1 \end{matrix}$$

$$\text{Cofactor of A} \quad \text{Aij} = \begin{bmatrix} 7 & -1 & 10 \\ 19 & 7 & -2 \\ 3 & 19 & 14 \end{bmatrix}$$

$$\text{Adj A} = \text{Aij}^{\text{T}}$$

$$\text{Adj A} = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj\ A$$

$$= \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$X = A^{-1} \times B$$

$$= \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$= \frac{1}{68}\begin{bmatrix} 1{,}05{,}000 + 19{,}000 + 12{,}000 \\ -15{,}000 + 7{,}000 + 76{,}000 \\ 15{,}000 - 2{,}000 + 56{,}000 \end{bmatrix}$$

$$= \frac{1}{68}\begin{bmatrix} 1{,}36{,}000 \\ 68{,}000 \\ 2{,}04{,}000 \end{bmatrix}$$

$$= \begin{bmatrix} 2{,}000 \\ 1{,}000 \\ 3{,}000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2{,}000 \\ 1{,}000 \\ 3{,}000 \end{bmatrix}$$

The price of the commodity A = Rs. 2,000

The price of the commodity B = Rs. 1,000

The price of the commodity C = Rs. 3,000

Example 1.25 Solve, by Cramer's rule, the system of equations $x_1 - x_2 = 3$, $2x_1 + 3x_2 + 4x_3 = 17$ and $x_2 + 2x_3 = 7$.

Solution: Given $x_1 - x_2 + 0x_3 = 3$,

$$2x_1 + 3x_2 + 4x_3 = 17$$

$$0x_1 + x_2 + 2x_3 = 7$$

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$

$$= 1(6 - 4) + 1(4 - 0) + 0$$

$$= 1(2) + 1(4) + 0$$

$$= 2 + 4 + 0$$

$$= 6 \neq 0$$

$$\Delta_{x_1} = \begin{vmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 17 & 4 \\ 7 & 2 \end{vmatrix} + 0\begin{vmatrix} 17 & 3 \\ 7 & 1 \end{vmatrix}$$

$$= 3(6 - 4) + 1(34 - 28) + 0$$

$$= 3(2) + 1(6) + 0$$

$$= 6 + 6 + 0$$

$$= 12$$

$$\Delta_{x_2} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 17 & 4 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 17 \\ 0 & 7 \end{vmatrix}$$

$$= 1(34 - 28) - 3(4 - 0) + 0$$

$$= 1(6) - 3(4) + 0$$

$$= 6 - 12 + 0$$

$$= -6$$

$$\Delta_{x_3} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 7 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 17 \\ 1 & 7 \end{vmatrix} + 1\begin{vmatrix} 2 & 17 \\ 0 & 7 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$

$$= 1(21 - 17) + 1(14 - 0) + 3(2 - 0)$$

$= 1(4) + 1(14) + 3(2)$

$= 4 + 14 + 6$

$= 24$

By Cramer's rule, we get

$x_1 = \frac{\Delta_{x_1}}{\Delta} = \frac{12}{6} = 2$

$x_2 = \frac{\Delta_{x_2}}{\Delta} = \frac{-6}{6} = -1$

$x_3 = \frac{\Delta_{x_3}}{\Delta} = \frac{24}{6} = 4$

Example 1.26 In a T20 match, Chennai Super Kings needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2 + bx + c$ with respect to a xy-coordinate system in the vertical plane and the ball traversed through the points (10,8), (20,16), (40,22), can you conclude that Chennai Super Kings won the match? Justify your answer. (All distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is (70,0).)

Solution: The path $y = ax^2 + bx + c$ passes through the points (10,8), (20,16), (40,22).

So, we get the system of equations

$100a + 10b + c = 8$

$400a + 20b + c = 16$

$1600a + 40b + c = 22$

To apply Cramer's rule, we find

$$\Delta = \begin{vmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 1600 & 40 & 1 \end{vmatrix}$$

$$= 100\begin{vmatrix} 20 & 1 \\ 40 & 1 \end{vmatrix} - 10\begin{vmatrix} 400 & 1 \\ 1600 & 1 \end{vmatrix} + 1\begin{vmatrix} 400 & 20 \\ 1600 & 40 \end{vmatrix}$$

$= 100(20 - 40) - 10(400 - 1600) + 1(16000 - 32000)$

$= 100(-20) - 10(-1200) + 1(-16000)$

$= -2000 + 12000 - 16000$

$= -18000 + 12000$

$= -6000$

$$\Delta_a = \begin{vmatrix} 8 & 10 & 1 \\ 16 & 20 & 1 \\ 22 & 40 & 1 \end{vmatrix}$$

$$= 8\begin{vmatrix} 20 & 1 \\ 40 & 1 \end{vmatrix} - 10\begin{vmatrix} 16 & 1 \\ 22 & 1 \end{vmatrix} + 1\begin{vmatrix} 16 & 20 \\ 22 & 40 \end{vmatrix}$$

$= 8(20 - 40) - 10(16 - 22) + 1(640 - 440)$

$= 8(-20) - 10(-6) + 1(200)$

$= -160 + 60 + 200$

$= -160 + 260$

$= 100$

$$\Delta_b = \begin{vmatrix} 100 & 8 & 1 \\ 400 & 16 & 1 \\ 1600 & 22 & 1 \end{vmatrix}$$

$$= 100\begin{vmatrix} 16 & 1 \\ 22 & 1 \end{vmatrix} - 8\begin{vmatrix} 400 & 1 \\ 1600 & 1 \end{vmatrix} + 1\begin{vmatrix} 400 & 16 \\ 1600 & 22 \end{vmatrix}$$

$= 100(16 - 22) - 8(400 - 1600) + 1(8800 - 25600)$

$= 100(-6) - 8(-1200) + 1(-16800)$

$= -600 + 9600 - 16800$

$= -17400 + 9600$

$= -7800$

$$\Delta_c = \begin{vmatrix} 100 & 10 & 8 \\ 400 & 20 & 16 \\ 1600 & 40 & 22 \end{vmatrix}$$

$$= 100\begin{vmatrix} 20 & 16 \\ 40 & 22 \end{vmatrix} - 10\begin{vmatrix} 400 & 16 \\ 1600 & 22 \end{vmatrix} + 8\begin{vmatrix} 400 & 20 \\ 1600 & 40 \end{vmatrix}$$

$=100(440-640)-10(8800-25600)$

$+8(16000-32000)$

$=100(-200)-10(-16800)+8(-16000)$

$=-20000+168000-128000$

$=168000-148000$

$=20000$

By Cramer's rule, we get

$a=\frac{\Delta_a}{\Delta}=\frac{100}{-6000}=-\frac{1}{60}$

$b=\frac{\Delta_b}{\Delta}=\frac{-7800}{-6000}=\frac{78}{60}=\frac{13}{10}$

$c=\frac{\Delta_c}{\Delta}=\frac{20000}{-6000}=-\frac{20}{6}=-\frac{10}{3}$

So, the equation of the path is $y=ax^2+bx+c$ becomes

$y=-\frac{1}{60}x^2+\frac{13}{10}x-\frac{10}{3}$

substituting the point (70,0)

When $x=70$, in

$y=-\frac{1}{60}x^2+\frac{13}{10}x-\frac{10}{3}$

$=-\frac{1}{60}\times 70\times 70+\frac{13}{10}\times 70-\frac{10}{3}$

$=-\frac{490}{6}+91-\frac{10}{3}$

$=-\frac{245}{3}+91-\frac{10}{3}$

$=-\frac{255}{3}+91$

$=-85+91$

$=6$

We get $y=6$. So, the ball went by 6 metres high over the boundary line and it is impossible for a fielder standing even just before the boundary line to jump and catch the ball. Hence the ball went for a super six and the Chennai Super Kings won the match.

EXERCISE 1.4

1. Solve the following systems of linear equations by Cramer's rule:

(i) $5x-2y+16=0, x+3y-7=0$

Solution: $5x-2y+16=0$

$x+3y-7=0$ gives

$5x-2y=-16$

$x+3y=7$

$\Delta=\begin{vmatrix}5 & -2\\ 1 & 3\end{vmatrix}$

$=15+2$

$=17\neq 0$

$\Delta_x=\begin{vmatrix}-16 & -2\\ 7 & 3\end{vmatrix}$

$=-48+14$

$=-34$

$\Delta_y=\begin{vmatrix}5 & -16\\ 1 & 7\end{vmatrix}$

$=35+16$

$=51$

By Cramer's rule, we get

$x=\frac{\Delta_x}{\Delta}=\frac{-34}{17}=-2$

$y=\frac{\Delta_y}{\Delta}=\frac{51}{17}=3$

ii) $\frac{3}{x}+2y=12, \frac{2}{x}+3y=13$

Solution: $\frac{3}{x}+2y=12$

$\frac{2}{x}+3y=13$

Let $\frac{1}{x} = a$, then equation is

$$3a + 2y = 12$$

$$2a + 3y = 13$$

$$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 9 - 4$$

$$= 5 \neq 0$$

$$\Delta_a = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix}$$

$$= 36 - 26$$

$$= 10$$

$$\Delta_y = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix}$$

$$= 39 - 24$$

$$= 15$$

By Cramer's rule, we get

$$a = \frac{\Delta_a}{\Delta} = \frac{10}{5} = 2$$

$$y = \frac{\Delta_y}{\Delta} = \frac{15}{5} = 3$$

$$a = 2 = \frac{1}{x} \ gives\ x = \frac{1}{2}$$

$$y = 3$$

(iii) $3x + 3y - z = 11, 2x - y + 2z = 9$

and $4x + 3y + 2z = 25$

Solution : $3x + 3y - z = 11$

$$2x - y + 2z = 9$$

$$4x + 3y + 2z = 25$$

To apply Cramer's rule, we find

$$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= 3(-2 - 6) - 3(4 - 8) - 1(6 + 4)$$

$$= 3(-8) - 3(-4) - 1(10)$$

$$= -24 + 12 - 10$$

$$= -34 + 12$$

$$= -22 \neq 0$$

$$\Delta_x = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix}$$

$$= 11\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 9 & 2 \\ 25 & 2 \end{vmatrix} - 1\begin{vmatrix} 9 & -1 \\ 25 & 3 \end{vmatrix}$$

$$= 11(-2 - 6) - 3(18 - 50) - 1(27 + 25)$$

$$= 11(-8) - 3(-32) - 1(52)$$

$$= -88 + 96 - 52$$

$$= -140 + 96$$

$$= -44$$

$$\Delta_y = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 9 & 2 \\ 25 & 2 \end{vmatrix} - 11\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 9 \\ 4 & 25 \end{vmatrix}$$

$$= 3(18 - 50) - 11(4 - 8) - 1(50 - 36)$$

$$= 3(-32) - 11(-4) - 1(14)$$

$$= -96 + 44 - 14$$

$$= -110 + 44$$

$$= -66$$

$$\Delta_z = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 9 \\ 3 & 25 \end{vmatrix} - 3\begin{vmatrix} 2 & 9 \\ 4 & 25 \end{vmatrix} + 11\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$

$= 3(-25 - 27) - 3(50 - 36) + 11(6 + 4)$

$= 3(-52) - 3(14) + 11(10)$

$= -156 - 42 + 110$

$= -198 + 110$

$= -88$

By Cramer's rule, we get

$x = \frac{\Delta_x}{\Delta} = \frac{-44}{-22} = 2$

$y = \frac{\Delta_y}{\Delta} = \frac{-66}{-22} = 3$

$z = \frac{\Delta_z}{\Delta} = \frac{-88}{-22} = 4$

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0$

$\frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

Solution: $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0$

$\frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0$

$\frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

Let $\frac{1}{x} = a, \frac{1}{y} = b$ and $\frac{1}{z} = c$

$3a - 4b - 2c = 1$

$a + 2b + c = 2$

$2a - 5b - 4c = -1$

To apply Cramer's rule, we find

$$\Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix}$$

$= 3\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$

$= 3(-8 + 5) + 4(-4 - 2) - 2(-5 - 4)$

$= 3(-3) + 4(-6) - 2(-9)$

$= -9 - 24 + 18$

$= -33 + 18$

$= -15 \neq 0$

$$\Delta_a = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix}$$

$= 1\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ -1 & -5 \end{vmatrix}$

$= 1(-8 + 5) + 4(-8 + 1) - 2(-10 + 2)$

$= 1(-3) + 4(-7) - 2(-8)$

$= -3 - 28 + 16$

$= -31 + 16$

$= -15$

$$\Delta_b = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix}$$

$= 3\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$

$= 3(-8 + 1) - 1(-4 - 2) - 2(-1 - 4)$

$= 3(-7) - 1(-6) - 2(-5)$

$= -21 + 6 + 10$

$= -21 + 16$

$= -5$

$$\Delta_c = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix}$$

$= 3\begin{vmatrix} 2 & 2 \\ -5 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$

$= 3(-2 + 10) + 4(-1 - 4) + 1(-5 - 4)$

$= 3(8) + 4(-5) + 1(-9)$

$= 24 - 20 - 9$

$= 24 - 29$

$= -5$

By Cramer's rule, we get

$$a = \frac{\Delta_a}{\Delta} = \frac{-15}{-15} = 1 = \frac{1}{x} \; gives \; x = 1$$

$$b = \frac{\Delta_b}{\Delta} = \frac{-5}{-15} = \frac{1}{3} = \frac{1}{y} \; gives \; y = 3$$

$$c = \frac{\Delta_c}{\Delta} = \frac{-5}{-15} = \frac{1}{3} = \frac{1}{z} \; gives \; z = 3$$

2. In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly? (Use Cramer's rule to solve the problem).

Solution: Let the number of correctly answered questions $= x$

Let the number of not correctly answered questions $= y$

Given mark for each correct answers =1

Mark for each wrong answers $= -\frac{1}{4}$

$\therefore x + y = 100$ and

$$x - \frac{y}{4} = 80 \Rightarrow \frac{4x - y}{4} = 80$$

$$4x - y = 320$$

Solving $x + y = 100$

$$4x - y = 320$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix}$$

$= -1 - 4$

$= -5 \neq 0$

$$\Delta_x = \begin{vmatrix} 100 & 1 \\ 320 & -1 \end{vmatrix}$$

$= -100 - 320$

$= -420$

$$\Delta_y = \begin{vmatrix} 1 & 100 \\ 4 & 320 \end{vmatrix}$$

$= 320 - 400$

$= -80$

By Cramer's rule, we get

$$x = \frac{\Delta_x}{\Delta} = \frac{-420}{-5} = 84$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-80}{-5} = 16$$

The student answered 84 questions correctly.

3. A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution ? (Use Cramer's rule to solve the problem).

Solution:

Let A be the solutions which is 50% acid has x litres and B be the solutions which is 25% acid has y litres.

$$\left[x \times 50\% = x \times \frac{50}{100} = \frac{x}{2}\right]$$

$$\left[y \times 25\% = y \times \frac{25}{100} = \frac{y}{4}\right]$$

$$\left[10 \times 40\% = 10 \times \frac{40}{100} = 4\right]$$

$\therefore x + y = 10$ and

$$\frac{x}{2} + \frac{y}{4} = 4$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{4} \end{vmatrix}$$

$= \frac{1}{4} - \frac{1}{2}$

$= \frac{2-4}{8}$

$= \frac{-2}{8}$

$= -\frac{1}{4} \neq 0$

$\Delta_x = \begin{vmatrix} 10 & 1 \\ 4 & \frac{1}{4} \end{vmatrix}$

$= \frac{10}{4} - 4$

$= \frac{10-16}{4}$

$= -\frac{6}{4}$

$\Delta_y = \begin{vmatrix} 1 & 10 \\ \frac{1}{2} & 4 \end{vmatrix}$

$= 4 - \frac{10}{2}$

$= 4 - 5$

$= -1$

By Cramer's rule, we get

$$x = \frac{\Delta_x}{\Delta} = \frac{-\frac{6}{4}}{-\frac{1}{4}} = \frac{6}{4} \times \frac{4}{1} = 6$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-1}{-\frac{1}{4}} = 1 \times \frac{4}{1} = 4$$

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself ? (Use Cramer's rule to solve the problem).

Solution:

Let the pump A can fill the tank in x minutes and the pump B can fill the tank in y minutes.

In 1 minute pump A can fill $\frac{1}{x}$ part

In 1 minute pump B can fill $\frac{1}{y}$ part

By the data $\frac{1}{x} + \frac{1}{y} = \frac{1}{10}$

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{30}$$

To solve let $\frac{1}{x} = a,\ \frac{1}{y} = b$

$$a + b = \frac{1}{10}$$

$$a - b = \frac{1}{30}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$= -1 - 1$

$= -2 \neq 0$

$\Delta_a = \begin{vmatrix} \frac{1}{10} & 1 \\ \frac{1}{30} & -1 \end{vmatrix}$

$= -\frac{1}{10} - \frac{1}{30}$

$= \frac{-3-1}{30}$

$= -\frac{4}{30}$

$= -\frac{2}{15}$

$\Delta_b = \begin{vmatrix} 1 & \frac{1}{10} \\ 1 & \frac{1}{30} \end{vmatrix}$

$= \frac{1}{30} - \frac{1}{10}$

$$= \frac{1-3}{30}$$

$$= -\frac{2}{30}$$

$$= -\frac{1}{15}$$

By Cramer's rule, we get

$$a = \frac{\Delta_a}{\Delta} = \frac{-\frac{2}{15}}{-2} = \frac{2}{15} \times \frac{1}{2} = \frac{1}{15}$$

$$b = \frac{\Delta_b}{\Delta} = \frac{-\frac{1}{15}}{-2} = \frac{1}{15} \times \frac{1}{2} = \frac{1}{30}$$

$$a = \frac{1}{15} = \frac{1}{x} \text{ gives } x = 15$$

$$b = \frac{1}{30} = \frac{1}{x} \text{ gives } y = 30$$

Hence the pump A can fill the tank in 15 minutes and the pump B can fill the tank in 30 minutes.

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is Rs.150. The cost of the two dosai, two idlies and four vadais is Rs.200. The cost of five dosai, four idlies and two vadais is Rs. 250. The family has Rs. 350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had?

Solution: Let the cost of a dosai = Rs. x

The cost of a idly = Rs. y

The cost of a vadai = Rs. z

Given $2x + 3y + 2z = 150$

$$2x + 2y + 4z = 200$$

$$5x + 4y + 2z = 250$$

To apply Cramer's rule, we find

$$\Delta = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & 4 \\ 5 & 4 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix}$$

$$= 2(4-16) - 3(4-20) + 2(8-10)$$

$$= 2(-12) - 3(-16) + 2(-2)$$

$$= -24 + 48 - 4$$

$$= -28 + 48$$

$$= 20 \neq 0$$

$$\Delta_x = \begin{vmatrix} 150 & 3 & 2 \\ 200 & 2 & 4 \\ 250 & 4 & 2 \end{vmatrix}$$

$$= 150\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 200 & 4 \\ 250 & 2 \end{vmatrix} + 2\begin{vmatrix} 200 & 2 \\ 250 & 4 \end{vmatrix}$$

$$= 150(4-16) - 3(400-1000)$$

$$+2(800-500)$$

$$= 150(-12) - 3(-600) + 2(300)$$

$$= -1800 + 1800 + 600$$

$$= -1800 + 2400$$

$$= 600$$

$$\Delta_y = \begin{vmatrix} 2 & 150 & 2 \\ 2 & 200 & 4 \\ 5 & 250 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 200 & 4 \\ 250 & 2 \end{vmatrix} - 150\begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 200 \\ 5 & 250 \end{vmatrix}$$

$$= 2(400-1000) - 150(4-20) + 2(500-1000)$$

$$= 2(-600) - 150(-16) + 2(-500)$$

$$= -1200 + 2400 - 1000$$

$$= -2200 + 2400$$

$$= 200$$

$$\Delta_z = \begin{vmatrix} 2 & 3 & 150 \\ 2 & 2 & 200 \\ 5 & 4 & 250 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 200 \\ 4 & 250 \end{vmatrix} - 3\begin{vmatrix} 2 & 200 \\ 5 & 250 \end{vmatrix} + 150\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix}$$

$= 2(500 - 800) - 3(500 - 1000) + 150(8 - 10)$

$= 2(-300) - 3(-500) + 150(-2)$

$= -600 + 1500 - 300$

$= -900 + 1500$

$= 600$

By Cramer's rule, we get

$x = \frac{\Delta_x}{\Delta} = \frac{600}{20} = 30$

$y = \frac{\Delta_y}{\Delta} = \frac{200}{20} = 10$

$z = \frac{\Delta_z}{\Delta} = \frac{600}{20} = 30$ that is

The cost of a dosai = Rs. 30

The cost of a idly = Rs. 10

The cost of a vadai = Rs. 30

The family ate 3 dosai and six idlies and six vadais.

$3x + 6y + 6z = 3(30) + 6(10) + 6(30)$

$= 90 + 60 + 180$

$= 330$ Rs.

The family has Rs. 350 so they are able to manage to pay the bill.

Example 1.27 Solve the following system of linear equations, by Gaussian elimination method: $4x + 3y + 6z = 25$,

$x + 5y + 7z = 13, 2x + 9y + z = 1$

Solution: Given

$$4x + 3y + 6z = 25$$

$$x + 5y + 7z = 13$$

$$2x + 9y + z = 1$$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 4 & 3 & 6 & 25 \\ 2 & 9 & 1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & -1 & -13 & -25 \end{bmatrix} \begin{matrix} R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 2R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & 0 & -199 & -398 \end{bmatrix} R_3 = 17R_3 - R_2$$

The equivalent system is written by using the echelon form: $x + 5y + 7z = 13$

$$-17y - 22z = -27$$

$$-199z = -398$$

We get, $199z = 398$

$z = \frac{398}{199}$

$= 2$

Substituting z = 2 in $-17y - 22z = -27$

$-17y - 22(2) = -27$

$-17y - 44 = -27$

$-17y = -27 + 44$

$-17y = 17$

$y = \frac{17}{-17} = -1$

Substituting z = 2 and y = −1 in

$$x + 5y + 7z = 13$$

$$x + 5(-1) + 7(2) = 13$$

$$x - 5 + 14 = 13$$

$$x + 9 = 13$$

$$x = 13 - 9$$

$$x = 4$$

So the solution is $x = 4, y = -1$ and $z = 2$

Example 1.28 The upward speed v (t) of a rocket at time t is approximated by $v(t) = at^2 + bt + c, 0 \leq t \leq 100$ where a, b and c are constants. It has been found that the speed at times t = 3. t = 6 and t = 9 seconds are respectively, 64, 133, and 208 miles per second respectively. Find the speed at time t =15 seconds. (Use Gaussian elimination method.)
Solution: $v(t) = at^2 + bt + c$

At t = 3, $v(3) = a(3)^2 + b(3) + c = 64$

$$a(9) + b(3) + c = 64$$

$$9a + 3b + c = 64 \quad \ldots\ldots(1)$$

At t = 6, $v(6) = a(6)^2 + b(6) + c = 133$

$$a(36) + b(6) + c = 133$$

$$36a + 6b + c = 133 \quad \ldots\ldots(2)$$

At t = 9, $v(9) = a(9)^2 + b(9) + c = 208$

$$a(81) + b(9) + c = 208$$

$$81a + 9b + c = 208 \quad \ldots\ldots(3)$$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 9 & 3 & 1 & 64 \\ 36 & 6 & 1 & 133 \\ 81 & 9 & 1 & 208 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 & 1 & 64 \\ 0 & -6 & -3 & -123 \\ 0 & -18 & -8 & -368 \end{bmatrix} \begin{matrix} R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 9R_1 \end{matrix}$$

$$= \begin{bmatrix} 9 & 3 & 1 & 64 \\ 0 & -6 & -3 & -123 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_3 = R_3 - 3R_2$$

The equivalent system is written by using the echelon form: $9a + 3b + c = 64$

$$-6b - 3c = -123$$

$$c = 1$$

Substituting c = 1 in $-6b - 3c = -123$

$$-6b - 3(1) = -123$$

$$-6b - 3 = -123$$

$$-6b = -123 + 3$$

$$-6b = -120$$

$$6b = 120$$

$$b = \frac{120}{6}$$

$$= 20$$

Substituting c = 1 and b = 20 in

$$9a + 3y + z = 64$$

$$9a + 3(20) + 1 = 64$$

$$9a + 60 + 1 = 64$$

$$9a + 61 = 64$$

$$9a = 64 - 61$$

$$9a = 3$$

$$a = \frac{3}{9}$$

$$= \frac{1}{3}$$

So substituting $a = \frac{1}{3}$, $b = 20$ and c =1 in

$$v(t) = at^2 + bt + c \quad \text{we get}$$

$$v(t) = \frac{1}{3}t^2 + 20t + 1$$

The speed at t = 15 minutes

$$v(15) = \frac{1}{3}(15)^2 + 20(15) + 1$$

$$= \frac{1}{3}(225) + 300 + 1$$

$$= 75 + 301$$

$$= 376.$$

EXERCISE 1.5

1. Solve the following systems of linear equations by Gaussian elimination method:

(i) $2x - 2y + 3z = 2,\ x + 2y - z = 3,$

$3x - y + 2z = 1$

Solution: $2x - 2y + 3z = 2,$

$$x + 2y - z = 3,$$

$$3x - y + 2z = 1$$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & 0 & -5 & -20 \end{bmatrix} R_3 = 6R_3 - 7R_2$$

The equivalent system is written by using the echelon form: $x + 2y - z = 3$

$$-6y + 5z = -4$$

$$-5z = -\,20$$

We get, $5z = 20$

$$z = \frac{20}{5}$$

$$= 4$$

Substituting z = 4 in $-6y + 5z = -4$

$$-6y + 5(4) = -4$$

$$-6y + 20 = -4$$

$$-6y = -4 - 20$$

$$-6y = -24$$

$$6y = 24$$

$$y = \frac{24}{6}$$

$$= 4$$

Substituting

z = 4 and y = 4 in $x + 2y - z = 3$

$$x + 2(4) - 4 = 3$$

$$x + 8 - 4 = 3$$

$$x + 4 = 3$$

$$x = 3 - 4$$

$$x = -1$$

So the solution is $x = -1, y = 4\ and\ z = 4$

(ii) $2x + 4y + 6z = 22, 3x + 8y + 5z = 27,$

$-x + y + 2z = 2$

Solution: $2x + 4y + 6z = 22$

$$3x + 8y + 5z = 27$$

$$-x + y + 2z = 2$$

(1) $is \div 2$, then

$$x + 2y + 3z = 11$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{bmatrix} \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 + R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 0 & 22 & 44 \end{bmatrix} R_3 = 2R_3 - 3R_2$$

The equivalent system is written by using the echelon form: $x + 2y + 3z = 11$

$$2y - 4z = -6$$
$$22z = 44$$

We get, $z = \frac{44}{22}$

$=2$

Substituting z = 2 in $2y - 4z = -6$

$$2y - 4(2) = -6$$
$$2y - 8 = -6$$
$$2y = -6 + 8$$
$$2y = 2$$

$y = \frac{2}{2}$

$=1$

Substituting

z = 2 and y = 1 in $x + 2y + 3z = 11$

$$x + 2(1) + 3(2) = 11$$
$$x + 2 + 6 = 11$$
$$x + 8 = 11$$
$$x = 11 - 8$$
$$x = 3$$

So the solution is $x = 3, y = 1 \; and \; z = 2$

2. If $ax^2 + bx + c$ is divided by x+3, x – 5 and x −1, the remainders are 21,61 and 9 ,and respectively. Find a, b and c. (Use Gaussian elimination method.)

Solution: Let $P(x) = ax^2 + bx + c$ is divided by x+3, x – 5 and x −1 then the remainder is $P(-3)$, $P(5)$ and $P(1)$.

From the data given,

$$P(-3) = a(-3)^2 + b(-3) + c = 21$$
$$9a - 3b + c = 21 \; \ldots..(1)$$
$$P(5) = a(5)^2 + b(5) + c = 61$$
$$25a + 5b + c = 61 \; \ldots..(2)$$
$$P(1) = a(1)^2 + b(1) + c = 9$$
$$a + b + c = 9 \; \ldots..(3)$$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ 1 & 1 & 1 & 9 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{bmatrix} R_{1\leftrightarrow}R_3$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{bmatrix} \begin{matrix} R_2 = R_2 - 25R_1 \\ R_3 = R_3 - 9R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 5 & 6 & 41 \\ 0 & 0 & -8 & -48 \end{bmatrix} \begin{matrix} R_2 = R_2 \div (-4) \\ R_3 = R_3 \div (-4) \end{matrix}$$

The equivalent system is written by using the echelon form: $a + b + c = 9$

$$5b + 6c = 41$$

$$-8c = -48$$

We get, $8c = \ 48$

$$c = \frac{48}{8}$$

$$=6$$

Substituting c = 6in $5b + 6c = 41$

$$5b + 6(6) = 41$$

$$5b + 36 = 41$$

$$5b = 41 - 36$$

$$5b = 5$$

$$b = \frac{5}{5} = 1$$

Substituting

c = 6 and b = 1 in $a + b + c = 9$

$$a + (1) + (6) = 9$$

$$a + 1 + 6 = 9$$

$$a + 7 = 9$$

$$a = 9 - 7$$

$$a = 2$$

So the solution is $a = 2, b = 1 \ and \ c = 6$

3. An amount of Rs.65,000 is invested in three bonds at the rates of 6%, 8%, and 9 % per annum respectively. The total annual income is Rs. 4,800. The income from the third bond is Rs. 600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

Solution:

Let the prices of each bond is Rs. x, y, z.

By the data, $x + y + z = 65{,}000$ ....(1)

$$\frac{6x}{100} + \frac{8y}{100} + \frac{9z}{100} = 4800 \ gives$$

$$6x + 8y + 9z = 4{,}80{,}000 \quad .....(2)$$

$$-\frac{8y}{100} + \frac{9z}{100} = 600 \ gives$$

$$-8y + 9z = 60{,}000 \quad .....(3)$$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 65{,}000 \\ 6 & 8 & 9 & 4{,}80{,}000 \\ 0 & -8 & 9 & 60{,}000 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 65{,}000 \\ 0 & 2 & 3 & 90.000 \\ 0 & -8 & 9 & 60{,}000 \end{bmatrix} R_2 = R_2 - 6R_1$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 65{,}000 \\ 0 & 2 & 3 & 90.000 \\ 0 & 0 & 21 & 4{,}20{,}000 \end{bmatrix} R_3 = R_3 + 4R_2$$

The equivalent system is written by using the echelon form: $x + y + z = 65{,}000$

$$2y + 3z = 90{,}000$$

$$21z = 4{,}20{,}000$$

We get, $z = \frac{4{,}20{,}000}{21}$

$$=20{,}000$$

Substituting z = 20,000 in $2y + 3z = 90{,}000$

$$2y + 3(20{,}000) = 90{,}000$$

$$2y + 60{,}000 = 90{,}000$$

$2y = 90{,}000 - 60{,}000$

$2y = 30{,}000$

$y = \frac{30{,}000}{2}$

$= 15{,}000$

Substituting

$z = 20{,}000$ and $y = 15{,}000$ in

$x + y + z = 65{,}000$

$x + 15{,}000 + 20{,}000 = 65{,}000$

$x + 35{,}000 = 65{,}000$

$x = 65{,}000 - 35{,}000$

$x = 30{,}000$

The prices of each bond is Rs.30,000

Rs.15,000and Rs.20,000 respectively.

4. A boy is walking along the path $y = ax^2 + bx + c$ through the points $(-6, 8), (-2, -12)$ and $(3, 8)$. He wants to meet his friend at $P(7, 60)$. Will he meet his friend? (Use Gaussian elimination method.)

Solution: $y = ax^2 + bx + c$

Substituting the given points $(-6, 8)$, $(-2, -12)$ and $(3, 8)$ in the above equation

we get $8 = a(-6)^2 + b(-6) + c$

$8 = 36a - 6b + c$

$36a - 6b + c = 8$ .....(1)

Similarly

$4a - 2b + c = -12$ .....(2)

$9a + 3b + c = 8$ .....(3)

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{bmatrix} \begin{matrix} R_2 = 9R_2 - R_1 \\ R_3 = 4R_3 - R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 36 & -6 & 1 & 8 \\ 0 & 6 & -4 & 58 \\ 0 & 6 & 1 & 8 \end{bmatrix} \begin{matrix} R_2 \div (-2) \\ R_3 \div (3) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 36 & -6 & 1 & 8 \\ 0 & 6 & -4 & 58 \\ 0 & 0 & -5 & 50 \end{bmatrix} R_3 = R_3 - R_2$$

The equivalent system is written by using the echelon form: $36a - 6b + c = 8$

$6b - 4c = 58$

$-5c = 50$

We get, $c = \frac{50}{-5}$

$= -10$

Substituting $c = -10$ in $6b - 4c = 58$

$6b - 4(-10) = 58$

$6b + 40 = 58$

$6b = 58 - 40$

$6b = 18$

$b = \frac{18}{6}$

$b = 3$

Substituting $c = -10$ and $b = 3$ in

$36a - 6b + c = 8$

$36a - 6(3) + (-10) = 8$

$36a - 18 - 10 = 8$

$$36\,a - 28 = 8$$

$$36\,a = 8 + 28$$

$$36\,a = 36$$

$$a = \frac{36}{36}$$

$$a = 1$$

Substituting $a = 1, b = 3$ and $c = -10$ in

$y = ax^2 + bx + c$ it becomes

$$y = x^2 + 3x - 10$$

substituting the point $P(7, 60)$

$$60 = (7)^2 + 3(7) - 10$$

$$60 = 49 + 21 - 10$$

$$60 = 70 - 10$$

$$60 = 60$$

So the boy meets his friend.

Example 1.29 Test for consistency of the following system of linear equations and if possible solve:

$x + 2y - z = 3,\ 3x - y + 2z = 1$

$x - 2y + 3z = 3$ and $x - y + z + 1 = 0$

Solution

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -4 & 4 & 0 \\ 0 & -3 & 2 & -4 \end{bmatrix} \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - R_1 \\ R_4 = R_4 - R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 4 & -4 & 0 \\ 0 & 3 & -2 & 4 \end{bmatrix} \begin{matrix} R_3 \times (-1) \\ R_4 \times (-1) \end{matrix}$$

$$\to$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & -8 & -32 \\ 0 & 0 & 4 & 16 \end{bmatrix} \begin{matrix} R_3 = 7R_3 + 4R_1 \\ R_4 = 4R_3 - 3R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & -8 & -32 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 = 2R_4 + R_3$$

There are three non-zero rows in the row-echelon form of $[AB]$. So, $\rho[AB]= \rho[A] = 3$

$\rho(AB) = \rho(A) = 3, n = 3$ The given equation is consistent, has unique solution.

$$-8z = -32$$

$$z = \frac{-32}{-8} = 4$$

sub z = 4 in

$$-7y + 5z = -8$$

$$-7y + 5(4) = -8$$

$$-7y + 20 = -8$$

$$-7y = -8 - 20$$

$$-7y = -28$$

$$7y = 28$$

$$y = \frac{28}{7} = 4$$

sub z = 4 and y = 4 in $x + 2y - z = 3$

$$x + 2(4) - 4 = 3$$

$$x + 8 - 4 = 3$$

$$x + 4 = 3$$

$$x = 3 - 4$$

$$x = -1$$

So the solution is $x = -1, y = 4$ and $z = 4$

Example 1.30

Test for consistency of the following system of linear equations and if possible solve:

$4x - 2y + 6z = 8,\ x + y - 3z = -1$

$15x - 3y + 9z = 21.$

Solution

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{bmatrix} \begin{matrix} R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 15R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 = R_3 - 3R_2$$

There are two non-zero rows in the row-echelon form of $[AB]$. So, $\rho[AB] = \rho[A] = 2$

$\rho(AB) = \rho(A) = 2, n = 3$ The given equation is consistent, has infinitely many solutions.

$$x + y - 3z = -1 \quad \ldots.(1)$$

$$-6y + 18z = 12 \quad \ldots.(2)$$

$$(2) \div -6$$

$$y - 3z = -2 \quad \ldots.(2)$$

To solve the equations let z = t,

then $\quad y - 3t = -2$

$$y = -2 + 3t \Rightarrow 3t - 2$$

substituting

$$z = t, y = 3t - 2 \ in \ x + y - 3z = -1$$

$$x + 3t - 2 - 3t = -1$$

$$x + 6t - 2 = -1$$

$$x = -1 - 6t + 2$$

$$x = 1 - 6t$$

So the solution

$$x = 1 - 6t, y = 3t - 1, z = t \ where \ t \in R$$

Example 1.31 Test for consistency of the following system of linear equations and if possible solve:

$x - y + z = -9,\ 2x - 2y + 2z = -18$

$3x - 3y + 3z + 27 = 0.$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 1 & -1 & 1 & -9 \\ 2 & -2 & 2 & -18 \\ 3 & -3 & 3 & -27 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$$

There is one non-zero row in the row-echelon form of $[AB]$. So, $\rho[AB] = \rho[A] = 1$

$\rho(AB) = \rho(A) = 1, n = 3$ The given equation is consistent, has infinitely many solutions.

$$x - y + z = -9$$

To solve the equations let $y = s, z = t,$

then $\quad x - s + t = -9$

$$x = -9 + s - t$$

So the solution

$x = -9 + s - t, y = s, z = t$ *where* $s, t \in R$

Example 1.32
Test for consistency of the following system of linear equations and if possible solve:

$x - y + z = -9,\ 2x - y + z = 4$

$3x - y + z = 6$ and $4x - y + 2z = 7$

Solution
Here the number of unknowns is 3.
The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -9 \\ 4 \\ 6 \\ 7 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 1 & -1 & 1 & -9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 2 & -2 & 33 \\ 0 & 3 & -2 & 43 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \\ R_4 = R_4 - 4R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 1 & -23 \end{bmatrix} \begin{matrix} R_3 = R_3 - 2R_2 \\ R_4 = R_4 - 3R_2 \end{matrix}$$

$$\to \begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -23 \\ 0 & 0 & 0 & -11 \end{bmatrix} R_{3 \leftrightarrow} R_4$$

$\rho(AB) = 4$ *and* $\rho(A) = 3$

$\rho(AB) \neq \rho(A)$ The given equation is

inconsistent, has **no solutions.**

Example 1.33
Find the condition on a,b and c so that the following system of linear equations has one parameter family of solutions:

$x + y + z = a,\ x + 2y + 3z = b$

$3x + 5y + 7z = c$

Solution:
Here the number of unknowns is 3.

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 1 & 1 & 1 & a \\ 1 & 2 & 3 & b \\ 3 & 5 & 7 & c \end{bmatrix}$$

$$\to$$

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b - a \\ 0 & 2 & 4 & c - 3a \end{bmatrix} \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b - a \\ 0 & 0 & 0 & c - 2b - a \end{bmatrix} R_3 = R_3 - 2R_2$$

In order that the system should have one parameter family of solutions, we must have $\rho(AB) = \rho(A) = 2$. So, the third row in the echelon form should be a zero row. So, $c - 2b - a = 0$ hence $c = 2b + a = 0$.

Example 1.34

Investigate for what values of λ and μ the system of linear equations $x + 2y + z = 7$, $x + y + \lambda z = \mu, x + 3y - 5z = 5$ has

(i) no solution (ii) a unique solution

(iii) an infinite number of solutions.

Solution:

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & \lambda \\ 1 & 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 7 \\ \mu \\ 5 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 1 & 2 & 1 & 7 \\ 1 & 1 & \lambda & \mu \\ 1 & 3 & -5 & 5 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -1 & \lambda - 1 & \mu - 7 \\ 0 & 1 & -6 & -2 \end{bmatrix} \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 2 & -6 & -2 \\ 0 & 0 & \lambda - 7 & \mu - 9 \end{bmatrix} R_3 = R_3 + R_2$$

(i) If $\lambda = 7$ and $\mu \neq 9$, then $\rho(AB) = 3$ *and*

$\rho(A) = 2. \rho(AB) \neq \rho(A)$

The given equation is **inconsistent**, has

nosolution.

(ii) If $\lambda \neq 7$ and μ has any value, then

$\rho(AB) = \rho(A) = 3, n = 3.$

The given equation is **consistent**,

has **unique solution.**

(iii) If $\lambda = 7$ and $\mu = 9$, then

$\rho(AB) = \rho(A) = 2, n < 3.$

The given equation is **consistent**,

has **infinite solutions**.

EXERCISE 1.6

1. Test for consistency and if possible, solve the following systems of equations by rank method.

(i) $x - y + 2z = 2,\ 2x + y + 4z = 7$

$4x - y + z = 4$

Solution:

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$,

where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 4 \\ 4 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 1 & -1 & 2 & 2 \\ 2 & 1 & 4 & 7 \\ 4 & -1 & 1 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & -7 & -4 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 4R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -7 & -7 \end{bmatrix} R_3 = R_3 - R_2$$

There are three non-zero rows in the row-echelon form of $[AB]$. So, $\rho[AB] = \rho[A] = 3$

$\rho(AB) = \rho(A) = 3, n = 3$ The given equation is consistent, has unique solution.

$-7z = -7 \Rightarrow 7z = 7$

$$z = \frac{7}{7} = 1$$

sub z = 1 in

3y + 0z = 3

3y = 3

$y = \frac{3}{3} = 1$

sub z = 1and y = 1 in $x - y + 2z = 2$

$x - 1 + 2(1) = 2$

$x - 1 + 2 = 2$

$x + 1 = 2$

$x = 2 - 1$

$x = 1$

So the solution is $x = 1. y = 1$, and $z = 1$.

(ii) $3x + y + z = 2,\ x - 3y + 2z = 1$

$7x - y + 4z = 5$

Solution:
Here the number of unknowns is 3.

The matrix form of the system is $AX = B$,

where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

The augmented matrix to echelon form,

we get $[AB] = \begin{bmatrix} 3 & 1 & 1 & 2 \\ 1 & -3 & 2 & 1 \\ 7 & -1 & 4 & 5 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & -3 & 2 & 1 \\ 3 & 1 & 1 & 2 \\ 7 & -1 & 4 & 5 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\rightarrow \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 20 & -10 & -2 \end{bmatrix} \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 7R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 = R_3 - 2R_2$$

There are two non-zero rows in the row-echelon form of $[AB]$. So, $\rho[AB] = \rho[A] = 2$

$\rho(AB) = \rho(A) = 2, n = 3$ The given equation is consistent, has infinitely many solutions.

$x - 3y + 2z = 1 \quad \ldots.(1)$

$10y - 5z = -1 \quad \ldots.(2)$

To solve the equations let z = t,

then $\quad 10y - 5t = -1$

$10\,y = -1 + 5t \Rightarrow 5t - 1$

$y = \frac{5t-1}{10}$

substituting

$z = t, y = \frac{5t-1}{10}$ $in\, x - 3y + 2z = 1$

$x - 3\left(\frac{5t-1}{10}\right) + 2t = 1$

$x = 1 - 2t + 3\left(\frac{5t-1}{10}\right)$

$= 1 - 2t + \frac{15t-3}{10}$

$= \frac{10-20t+15t-3}{10}$

$x = \frac{7-5t}{10}$

Solution:

$x = \frac{7-5t}{10}, y = \frac{5t-1}{10}$ and z = t where $t \in R$

(iii) $2x + 2y + z = 5,\ x - y + z = 1$

$3x + y + 2z = 4$

Solution:
Here the number of unknowns is 3.

The matrix form of the system is $AX = B$,

where

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{bmatrix} R_{1\leftrightarrow}R_2$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 1 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix} R_3 = R_3 - R_2$$

$\rho(AB) = 3$ *and* $\rho(A) = 2$

$\rho(AB) \neq \rho(A)$ The given equation is **inconsistent**, has **no solutions.**

(iv) $2x - y + z = 2,\ 6x - 3y + 3z = 6$

$4x - 2y + 2z = 4$

Solution:

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$,

where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 6 & -3 & 3 \\ 4 & -2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 2 & -1 & 1 & 2 \\ 6 & -3 & 3 & 6 \\ 4 & -2 & 2 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 1 & 2 \\ 2 & -1 & 1 & 2 \\ 2 & -1 & 1 & 2 \end{bmatrix} \begin{matrix} R_2 = R_2 \div 3 \\ R_3 = R_3 \div 2 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

There is one non-zero row in the row-echelon form of $[AB]$. So, $\rho[AB] = \rho[A] = 1$

$\rho(AB) = \rho(A) = 1, n = 3$ The given equation is consistent, has infinitely many solutions.

$$2x - y + z = 2$$

To solve the equations let $y = s, z = t$,

$$2x - s + t = 2$$

$$2x = 2 + s - t$$

$$x = \frac{2+s-t}{2}$$

So the solution

$x = \frac{2+s-t}{2}, y = s, z = t$ *where* $s, t \in R$

2. Find the value of k for which the equations $kx - 2y + z = 1$, $x - 2ky + z = -2, x - 2y + kz = 1$, have (i) no solution (ii) unique solution

(iii) infinitely many solution.

Solution: Here the number of unknowns is 3.The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 1 & -2 & k & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & k & 1 \\ 1 & -2k & 1 & -2 \\ k & -2 & 1 & 1 \end{bmatrix} R_{1} \leftrightarrow R_{3}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & k & 1 \\ 0 & 2-2k & 1-k & -3 \\ 0 & -2+2k & 1-k^2 & 1-k \end{bmatrix} \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - kR_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & k & 1 \\ 0 & 2-2k & 1-k & -3 \\ 0 & 0 & 2-k-k^2 & -2-k \end{bmatrix} R_3 = R_3 + R_2$$

$$\rightarrow \begin{bmatrix} 1 & -2 & k & 1 \\ 0 & 2(1-k) & (1-k) & -3 \\ 0 & 0 & (2+k)(1-k) & -(2+k) \end{bmatrix}$$

(i)When k =1, $\rho(AB) = 3$ *and* $\rho(A) = 2$

$\rho(AB) \neq \rho(A)$ The given equation is

inconsistent, has **no solutions.**

(ii) $k \neq 1$ *and* $k \neq -2$ $\rho(AB) = \rho(A) = 3$

and $n = 3$ The given equation is

consistent, has unique solution.

(iii) $k = -2, \rho(AB) = \rho(A) = 2$

and $n = 3$ The given equation is

consistent, has infinitely many solutions.

3. Investigate the values of λ and μ the system of linear equations $2x + 3y + 5z = 9, 7x + 3y - 5z = 8,$ $2x + 3y + \lambda z = \mu$, have (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Solution: Here the number of unknowns is 3. The matrix form of the system is $AX = B$ where

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -5 \\ 2 & 3 & \lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -45 & -47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix} \begin{matrix} R_2 = 2R_2 - 7R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

(i)When λ =5, $\rho(AB) = 3$ *and* $\rho(A) = 2$

$\rho(AB) \neq \rho(A)$ The given equation is

inconsistent, has **no solutions.**

(ii) $\lambda \neq 5$ *and* $\mu \neq 9$ $\rho(AB) = \rho(A) = 3$

and $n = 3$ The given equation is

consistent, has **unique solution.**

(iii) $\lambda = 5$ *and* $\mu = 9$, $\rho(AB) = \rho(A) = 2$

and $n = 3$ The given equation is

consistent, has **infinitely many solutions.**

Example 1.35 Solve the following system:
$x + 2y + 3z = 0, 3x + 4y + 4z = 0,$
$7x + 10y + 12z = 0.$

Solution

Here the number of equations is equal to the number of unknowns.

Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 4 & 4 & 0 \\ 7 & 10 & 12 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & -4 & -9 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 7R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} R_3 = R_3 - 2R_2$$

So, $\rho(AB) = \rho(A) = 3$ and $n = 3$.

Hence, the system has a unique solution.

Since $x = 0, y = 0, z = 0$, is always a solution of the homogeneous system, the only solution is the trivial solution $x = 0, y = 0, z = 0$.

Example 1.36 Solve the following system:

$x + 3y - 2z = 0, 2x - y + 4z = 0,$

$x - 11y + 14z = 0.$

Solution

Here the number of equations is equal to the number of unknowns.

Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & -1 & 4 & 0 \\ 1 & -11 & 14 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & -14 & 16 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 = R_3 - 2R_2$$

So, $\rho(AB) = \rho(A) = 2$ and $n = 3$.

Hence, the system has a one parameter family of solutions. Writing the equations using the echelon form, we get

$x + 3y - 2z = 0$ and $-7y + 8z = 0$

To solve the equations let z = t,

then $-7y + 8t = 0$

$$-7y = -8t$$

$$7y = 8t$$

$$y = \frac{8t}{7}$$

substituting

$$z = t, y = \frac{8t}{7} \; in \; x + 3y - 2z = 0$$

$$x + 3\left(\frac{8t}{7}\right) - 2t = 0$$

$$x + \frac{24t}{7} - 2t = 0$$

$$x = 2t - \frac{24t}{7}$$

$$x = \frac{14t - 24t}{7}$$

$$x = -\frac{10t}{7}$$

So the solution is

$$x = -\frac{10t}{7}, y = \frac{8t}{7} \text{ and } z = t \text{ where } t \in R$$

Example 1.37 Solve the following system:

$x + y - 2z = 0, 2x - 3y + z = 0,$

$3x - 7y + 10z = 0, 6x - 9y + 10z = 0.$

Solution

Here the number of equations is equal to the number of unknowns.

Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 2 & -3 & 1 & 0 \\ 3 & -7 & 10 & 0 \\ 6 & -9 & 10 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -10 & 16 & 0 \\ 0 & -15 & 22 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \\ R_4 = R_4 - 6R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix} \begin{matrix} R_3 = R_3 - 2R_2 \\ R_4 = R_4 - 3R_2 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 = 6R_4 - 7R_3$$

So, $\rho(AB) = \rho(A) = 3$ and $n = 3$.

Hence, the system has a unique solution.

Since $x = 0, y = 0, z = 0$, is always a solution of the homogeneous system, the only solution is the trivial solution

$x = 0, y = 0, z = 0$.

Example 1.38 Determine the values of λ for which the following system of equations

$(3\lambda - 8)x + 3y + 3z = 0,$

$3x + (3\lambda - 8)y + 3z = 0,$

$3x + 3y + (3\lambda - 8)z = 0.$

Solution

Here the number of unknowns is 3. So, if the system is consistent and has a non-trivial solution, then the rank of the coefficient matrix is equal to the rank of the augmented matrix and is less than 3. So the determinant of the coefficient matrix should be 0.

Hence we get

$$\begin{vmatrix} (3\lambda-8) & 3 & 3 \\ 3 & (3\lambda-8) & 3 \\ 3 & 3 & (3\lambda-8) \end{vmatrix} = 0$$

$R_1 = R_1 + R_2 + R_3$

$$\rightarrow \begin{vmatrix} 3\lambda-2 & 3\lambda-2 & 3\lambda-2 \\ 3 & (3\lambda-8) & 3 \\ 3 & 3 & (3\lambda-8) \end{vmatrix} = 0$$

Taking $(3\lambda - 2)$from R_1

$$\rightarrow (3\lambda-2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & (3\lambda-8) & 3 \\ 3 & 3 & (3\lambda-8) \end{vmatrix} = 0$$

$C_2 = C_1$ and $C_3 = C_1$

$$\rightarrow (3\lambda-2)\begin{vmatrix} 1 & 0 & 1 \\ 3 & 3\lambda-11 & 0 \\ 3 & 0 & 3\lambda-11 \end{vmatrix} = 0$$

$$(3\lambda-2)(3\lambda-11)^2 = 0$$

$3\lambda - 2 = 0$ and $3\lambda - 11 = 0$

Gives

$$3\lambda = 2 \Rightarrow \lambda = \frac{2}{3} \text{ and } 3\lambda = 11 \Rightarrow \lambda = \frac{11}{3}$$

Example 1.39 By using Gaussian elimination method, balance the chemical reaction equation: $C_5 + H_8 \rightarrow CO_2 + H_2O$

Solution

We are searching for positive integers

x_1, x_2, x_3 and x_4 such that

$x_1\, C_5 + x_2H_8 \rightarrow x_3CO_2 + x_4H_2O$ ....(1)

The number of carbon atoms on the left-hand side of (1) should be equal to the number of carbon atoms on the right-hand side of (1). So we get a linear homogenous equation

$5x_1 = x_3$ gives $5x_1 - x_3 = 0$ ....(2)

Similarly, considering hydrogen and oxygen atoms, we get respectively,

$8x_2 = 2x_4$ gives $4x_2 - x_4 = 0$ ....(3)

$2x_2 = 2x_3 + x_4$ gives

$2x_2 - 2x_3 - x_4 = 0$ ....(4)

Equations (2), (3), and (4) constitute a homogeneous system of linear equations in four unknowns. The augmented matrix is

$$[AB] = \begin{bmatrix} 5 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 5 & 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & -5 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} \mathrm{R}_2 = 5\mathrm{R}_2 - 4\mathrm{R}_1$$

$$\rightarrow \begin{bmatrix} 5 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 4 & -5 & 0 \end{bmatrix} R_2 \leftrightarrow R_3$$

$\rho(AB) = \rho(A) = 3, n = 4$ The given equation is consistent, has infinitely many solutions. Writing the equations using the echelon form, we get

$5x_1 - x_3 = 0, 2x_2 - 2x_3 - x_4 = 0$ and

$4x_3 - 5x_4 = 0.$

Substituting $x_4 = t$ in $4x_3 - 5x_4 = 0$

$$4x_3 - 5t = 0$$

$$4x_3 = 5t$$

$$x_3 = \frac{5t}{4}$$

Substituting $x_4 = t$ and $x_3 = \frac{5t}{4}$ in

$$2x_2 - 2x_3 - x_4 = 0$$

$$2x_2 - 2\left(\frac{5t}{4}\right) - t = 0$$

$$2x_2 - \frac{5t}{2} - t = 0$$

$$2x_2 = \frac{5t}{2} + t$$

$$= \frac{5t+2t}{2}$$

$$= \frac{7t}{2}$$

$$x_2 = \frac{7t}{4}$$

Substituting $x_3 = \frac{5t}{4}$ in

$$5x_1 - x_3 = 0$$

$$5x_1 - \frac{5t}{4} = 0$$

$$5x_1 = \frac{5t}{4} \text{ gives}$$

$$x_1 = \frac{t}{4}$$

So, $x_1 = \frac{t}{4}, x_2 = \frac{7t}{4}, x_3 = \frac{5t}{4}$ and $x_4 = t$

Let us choose $t = 4$. Then

$x_1 = 1, x_2 = 7, x_3 = 5$ and $x_4 = 4$

So the balanced equation is

$$C_5 + 7H_8 \rightarrow 5CO_2 + 4H_2O.$$

Example 1.40 If the system of equations

$px + by + cz = 0, ax + qy + cz = 0$ and

$ax + by + rz = 0$, has a non-trivial solution and $p \neq a$, $q \neq b$ and $r \neq c$, prove that

$$\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

Solution

The system $px + by + cz = 0$, $ax + qy + cz = 0$ and $ax + by + rz = 0$ has a non-trivial solution. So we have

$$\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

$$R_2 = R_2 - R_1 , R_3 = R_3 - R_1$$

$$\begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix} = 0$$

$$\begin{vmatrix} p & b & c \\ -(p-a) & q-b & 0 \\ -(p-a) & 0 & r-c \end{vmatrix} = 0$$

Dividing C_1 by $(p-a)$, C_2 by $(q-b)$ and C_3 by $(r-c)$

$$\begin{vmatrix} \frac{p}{p-a} & \frac{b}{q-b} & \frac{c}{r-c} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$\frac{p}{p-a}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \frac{b}{q-b}\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} + \frac{c}{r-c}\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = 0$$

$$\frac{p}{p-a}(1-0) - \frac{b}{q-b}(-1-0) + \frac{c}{r-c}(0+1) = 0$$

$$\frac{p}{p-a}(1) - \frac{b}{q-b}(-1) + \frac{c}{r-c}(1) = 0$$

$$\frac{p}{p-a}+\frac{b}{q-b}+\frac{c}{r-c}=0$$

$$\frac{p}{p-a}+\frac{q-(q-b)}{q-b}+\frac{r-(r-c)}{r-c}=0$$

$$\frac{p}{p-a}+\frac{q}{q-b}-\frac{q-b}{q-b}+\frac{r}{r-c}-\frac{r-c}{r-c}=0$$

$$\frac{p}{p-a}+\frac{q}{q-b}-1+\frac{r}{r-c}-1=0$$

$$\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}-2=0$$

$\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2$. Hence proved.

EXERCISE 1.7

1. Solve the following system of homogenous equations.

(i) $3x+2y+7z=0, 4x-3y-2z=0$

$5x+9y+23z=0,$

Solution

Here the number of equations is equal to the number of unknowns.

Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix} 3 & 2 & 7 & 0 \\ 4 & -3 & -2 & 0 \\ 5 & 9 & 23 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 2 & 7 & 0 \\ 0 & -17 & -34 & 0 \\ 0 & 17 & 34 & 0 \end{bmatrix} \begin{matrix} R_2=3R_2-4R_1 \\ R_3=3R_3-5R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 3 & 2 & 7 & 0 \\ 0 & -17 & -34 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{R}_3=\mathrm{R}_3+\mathrm{R}_2$$

So, $\rho(AB)= \rho(A)=2$ and $n=3$.

Hence, the system has a one parameter family of solutions. Writing the equations using the echelon form, we get

$3x+2y+7z=0$ and $-17y-34z=0$

To solve the equations let z = t,

then $-17y-34t=0$

$$-17y=34t$$

$$y=-\frac{34t}{17}$$

$$y=-2t$$

Substituting $z=t$ and $y=-2t$ in

$$3x+2y+7z=0$$

We get $3x+2(-2t)+7t=0$

$$3x-4t+7t=0$$

$$3x+3t=0$$

$$3x=-3t$$

$$x=-\frac{3t}{3}$$

$$x=-t$$

So the solution is

$x=-t, y=-2t$ and z = t where $t \in R$

(ii) $2x+3y-z=0, x-y-2z=0$

$3x+y+3z=0,$

Solution

Here the number of equations is equal to the number of unknowns. Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix} 2 & 3 & -1 & 0 \\ 1 & -1 & -2 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -2 & 0 \\ 2 & 3 & -1 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 4 & 9 & 0 \end{bmatrix} \begin{matrix} R_2=R_2-2R_1 \\ R_3=R_3-3R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 33 & 0 \end{bmatrix} R_3 = 5R_3 - 4R_2$$

So, $\rho(AB) = \rho(A) = 3$ and $n = 3$.

Hence, the system has a unique solution.

Since $x = 0, y = 0, z = 0$, is always a solution of the homogeneous system, the only solution is the trivial solution $x = 0, y = 0, z = 0$.

2. Determine the values of λ for which the following system of equations $x + y + 3z = 0, \quad 4x + 3y + \lambda z = 0,$ $2x + y + 2z = 0$,has (i) a unique solution (ii) a non-trivial solution.

Solution :Here the number of equations is equal to the number of unknowns. Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & \lambda - 12 & 0 \\ 0 & -1 & -4 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 2R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & \lambda - 12 & 0 \\ 0 & 0 & \lambda - 8 & 0 \end{bmatrix} R_3 = R_3 - R_2$$

(i) $\lambda \neq 8, \ \rho(AB) = \rho(A) = 3$ and $n = 3$

The given equation is **consistent**, has **unique solution.**

(ii) $\lambda = 8, \ \rho(AB) = \rho(A) = 2$ and $n = 3$

The given equation is **consistent**, has a non-trivial solution.

3. By using Gaussian elimination method, balance the chemical reaction equation:

$C_2H_6 + O_2 \rightarrow H_2O + CO_2$.

Solution

We are searching for positive integers x_1, x_2, x_3 and x_4 such that

$x_1\, C_2H_6 + x_2O_2 \rightarrow x_3H_2O + x_4CO_2$. ....(1)

The number of carbon atoms on the left-hand side of (1) should be equal to the number of carbon atoms on the right-hand side of (1). So we get a linear homogenous equation

$2x_1 = x_4$ gives $2x_1 - x_4 = 0$ ....(2)

Similarly, considering hydrogen and oxygen atoms, we get respectively,

$6x_1 = 2x_3$ gives $6x_1 - 2x_3 = 0$ ....(3)

$2x_2 = x_3 + 2x_4$ gives

$2x_2 - x_3 - 2x_4 = 0$ ....(4)

Equations (2), (3), and (4) constitute a homogeneous system of linear equations in four unknowns. The augmented matrix is

$$[AB] = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{bmatrix} R_2 = 2R_2 - 3R_1$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 0 & 0 & -2 & 3 & 0 \end{bmatrix} R_2 \leftrightarrow R_3$$

$\rho(AB) = \rho(A) = 3, n = 4$ The given equation is consistent, has infinitely many solutions. Writing the equations using the echelon form, we get

$-2x_3 + 3x_4 = 0, 2x_2 - x_3 - 2x_4 = 0$ and $2x_1 - x_4 = 0.$

Substituting $x_4 = t$ in $2x_1 - x_4 = 0$

$$2x_1 - t = 0$$

$$2x_1 = t$$

$$x_1 = \frac{t}{2}$$

Substituting $x_4 = t$ in $-2x_3 + 3x_4 = 0$

$$-2x_3 + 3t = 0$$

$$-2x_3 = -3t$$

$$2x_3 = 3t$$

$$x_3 = \frac{3t}{2}$$

Substituting $x_1 = \frac{t}{2}, x_3 = \frac{3t}{2}$ and $x_4 = t$ in

$$2x_2 - x_3 - 2x_4 = 0$$

$$2x_2 - \frac{3t}{2} - 2t = 0$$

$$2x_2 = \frac{3t}{2} + 2t$$

$$2x_2 = \frac{3t+4t}{2}$$

$$2x_2 = \frac{7t}{2}$$

$$x_2 = \frac{7t}{4}$$

So, $x_1 = \frac{t}{2}, x_2 = \frac{7t}{4}, x_3 = \frac{3t}{2}$ and $x_4 = t$

Let us choose $t = 4$. Then

$x_1 = 2, x_2 = 7, x_3 = 6$ and $x_4 = 4$

So the balanced equation is

$2\,C_2H_6 + 7O_2 \rightarrow 6H_2O + 4CO_2$.

EXERCISE 1.8

Choose the Correct answer:

1. If $|adj(adjA)| = |A|^9$ then the order of the square matrix A is . . . . .
   (1) 3 **(2) 4** (3) 2 (4) 5

2. If A is a 3×3 non-singular matrix such that A $A^T = A^T A$ and B $= A^{-1}A^T$, then $BB^T = \ldots\ldots$
   (1) A (2) B **(3) I** (4) B^T

3. If A $= \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, B = adj A and C = 3A, the $\frac{|adjB|}{|C|} = \ldots\ldots$
   (1) $\frac{1}{3}$ **(2) $\frac{1}{9}$** (3) $\frac{1}{4}$ (4) 1

4. If A $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then $A = \ldots\ldots$
   (1) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$
   (3) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

5. If A $= \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$, then 9I − A =
   (1) A^{-1} (2) $\frac{A^{-1}}{2}$ (3) $3A^{-1}$ **(4) $2\,A^{-1}$**

6. If $A = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ then

   | adj (AB) | =
   (1) −40 **(2) −80** (3) −60 (4) −20

7. If P $= \begin{bmatrix} 1 & x & 0 \\ 1 & 3 & 0 \\ 2 & 4 & -2 \end{bmatrix}$ is the adjoint of 3×3 matrix A and | A | = 4, then x is
   (1) 15 (2) 12 (3) 14 **(4) 11**

8. If A $= \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ and

$A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the value of a_{23} is . . . . .

(1) 0 (2) -2 (3) -3 **(4) -1**

9. If A, B and C are invertible matrices of some order, then which one of the following is not true?

(1) adj A = | A| $A^{-1}1$

(2) adj (AB) = adj (A)adj (B)

(3) det $A^{-1} = (detA)^{-1}$

(4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

10. If $(AB)^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, then $B^{-1} =$ . . . .

(1) $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$ (2) $\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$

(3) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

11. If A^TA^{-1} is symmetric, then $A^2 =$. . . . . .

(1) A^{-1} **(2) $(A^T)^2$**

(3) A^T (4) $(A^{-1})^2$

12. If A is a non-singular matrix such that $A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$, then $(A^T)^{-1} =$ . . . .

(1) $\begin{bmatrix} -5 & 3 \\ 2 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$

(3) $\begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$ **(4) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$**

13. If A = $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$ and $A^T = A^{-1}$, then the value of x is . . . . .

(1) $-\frac{4}{5}$ (2) $-\frac{3}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

14. If A = $\begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$ and AB = I, then B= . . . .

(1) $\left(cos^2\frac{\theta}{2}\right)A$ **(2) $\left(cos^2\frac{\theta}{2}\right)A^T$**

(3) $(cos^2\theta)I$ (4) $\left(sin^2\frac{\theta}{2}\right)A$

15. If A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $A(adjA) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then k is . . . .

(1) 0 (2) sinθ (3) cosθ **(4) 1**

16. If A = $\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that A^{-1}, then λ is . . . . .

(1) 17 (2) 14 **(3) 19** (4) 21

17. If adj A = $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and adjB = $\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ then adj(AB) is . . . . .

(1) $\begin{bmatrix} -7 & -1 \\ 7 & -9 \end{bmatrix}$ **(2) $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$**

(3) $\begin{bmatrix} -7 & 7 \\ -1 & -9 \end{bmatrix}$ (4) $\begin{bmatrix} -6 & -2 \\ 5 & -10 \end{bmatrix}$

18. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$ is . . . . . .

(1) 1 (2) 2 (3) 4 (4) 3

19. If $x^ay^b = e^m, x^cy^d = e^n, \Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}, \Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ then

the values of x and y are respectively

(1) $e^{\Delta_2/\Delta_1}, e^{\Delta_3/\Delta_1}$

(2) $\log(\Delta_1/\Delta_3), \log(\Delta_2/\Delta_3)$

(3) $\log(\Delta_2/\Delta_1), \log(\Delta_3/\Delta_1)$

(4) $\boldsymbol{e^{\Delta_1/\Delta_3}, e^{\Delta_2/\Delta_3}}$

20. Which of the following is/are correct?

(i) Adjoint of a symmetric matrix is also a symmetric matrix.

(ii) Adjoint of a diagonal matrix is also a diagonal matrix.

(iii) If A is a square matrix of order n and λ is a scalar, then $\text{adj}(\lambda A) = \lambda^n adj(A)$.

(iv) A(adjA)= (adjA)= |A | I

(1) Only (i) (2) (ii) and (iii)

(3) (iii) and (iv) **(4) (i), (ii) and (iv)**

21. If $\rho(A) = (AB)$, then the system $AX=B$ of linear equations is

(1) consistent and has a unique solution

(2) consistent

(3) consistent and has infinitely many solution

(4) inconsistent

22. If $0 \leq \theta \leq \pi$ and the system of equations $x + (\sin\theta)y - (\cos\theta)z = 0$, $(\cos\theta)x - y + z = 0$,$(\sin\theta)x + y - z = 0$ has a non-trivial solution then θ is ....

(1) $\frac{2\pi}{3}$ (2) $\frac{3\pi}{4}$ (3) $\frac{5\pi}{6}$ **(4)** $\boldsymbol{\frac{\pi}{4}}$

23. The augmented matrix of a system of linear equations is $\begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & \lambda-7 & \mu+5 \end{bmatrix}$ The system has infinitely many solutions if ......

(1) $= 7, \mu \neq -5$, (2) $\lambda = -7, \mu = 5$,

(3) $\lambda \neq 7, \mu \neq -5$, **(4)** $\boldsymbol{\lambda = 7, \mu = -5}$

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and $4B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & x \\ -1 & 1 & 3 \end{bmatrix}$. If B is the inverse of A, then the value of x is ......

(1) 2 (2) 4 (3) 3 **(4) 1**

25. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ then $\text{adj}(adjA)$ is ....

(1) $\begin{bmatrix} \mathbf{3} & \mathbf{-3} & \mathbf{4} \\ \mathbf{2} & \mathbf{-3} & \mathbf{4} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$ (2) $\begin{bmatrix} 6 & -6 & 8 \\ 4 & -6 & 8 \\ 0 & -2 & 2 \end{bmatrix}$

(3) $\begin{bmatrix} -3 & 3 & -4 \\ -2 & 3 & -4 \\ 0 & 1 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 1 \\ 2 & -3 & 4 \end{bmatrix}$

Formulae:

1. For 2x 2 matrix, adj A is obtained by

i) Interchange of main diagonal elements

ii) Change the sign of other elements.

2. Cofactor Matrix of A = A_{ij}

3. Adjoint Matrix of A = Aij^T

4. $A^{-1} = \frac{1}{|A|} Adj\ A$

5. A (adj A) = (adj A) A = | A | I_n

6. $A = \pm\frac{1}{\sqrt{adj\ A}} adj(adj\ A)$ and

$A^{-1} = \pm\frac{1}{\sqrt{adj\ A}} (adj\ A)$

7. $(A^T)^{-1} = (A^{-1})^T$

8. $(AB)^{-1} = B^{-1} A^{-1}$ and $(A^{-1})^{-1} = A$

9. If the Matrix is in Echelon form, then the number of non zero rows is the rank of the matrix and it is denoted by $\rho(A)$.

10. A square matrix A is called orthogonal

if $AA^T = A^TA = I$

11. A is called orthogonal if and only if

A is non singular and $A^{-1} = A^T$

12. $\cos 2x = cos^2x - sin^2x$ and

$\sin 2x = 2\sin x \cos x$

14. (i) $|A^{-1}| = \frac{1}{|A|}$ (ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$, λ is a scalar.

15. Methods to solve the system of linear

equations A X B =

(i) By matrix inversion $X = A^{-1}B$

(ii) By Cramer's rule if $\Delta \neq 0$

$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}$ and $z = \frac{\Delta_z}{\Delta}$

(iii) By Gaussian elimination method

16. If $\rho(AB) = \rho(A)$ then the given equation

is consistent.

17. If $\rho(AB) \neq \rho(A)$ then the given equation

is inconsistent.

18. If $\rho(AB) = \rho(A) = n$, the number of

unknowns then the given equation is

consistent and has unique solution.

19. If $\rho(AB) = \rho(A) \neq n$, then the given

equation is consistent and has

infinitely many solutions.

20. The homogenous system of linear

equations AX = 0

(i) has the trivial solution, if $|A| \neq 0$.

(ii) has a non trivial solution, if $|A| = 0$.